Centralized control scheme improves the stability margin but this strategy may lead to undesirable effects such as communication delay, data losing, switching network, etc. Therefore, it is necessary to study the whole traffic flow as interactions between several inter-connected vehicular platoons.

In this paper, internal and string stability were studied for both inter-platoon and intra-platoon networks. The main contributions of this paper are: 1. presenting a new approach to decouple the infinite dimension of closed-loop dynamics of heterogeneous traffic flow, 2. inter-platoon and intra-platoon string stability analysis of traffic flow, 3. considering both communication and parasitic delays in stability analysis of traffic flow with infinite dimension. The rest of this paper is organized as follows. In section 2, the problem is introduced. In section 3, internal stability of inter-platoon and intra-platoon were studied. In section 4, the string stability problem is investigated for traffic flow. In section 5, simulation results are provided to show the effectiveness of the proposed approaches. Finally, this paper is concluded in section 6.

2. PROBLEM DESCRIPTION

In Figure 1, the traffic flow is considered as the combination of infinite numbers of inter-connected cooperative heterogeneous vehicular platoons. $D_{i,i-1}$ illustrates the inter-platoon spacing and $d_{i,i-1}$ is the intra-platoon spacing. These assumptions are considered for this work: 1. The traffic flow consists of inter-connected heterogeneous vehicular platoons, 2. Each platoon consists of N following and one lead vehicles, 3. The communication topology in the whole traffic flow is unidirectional.
The dynamics of vehicle i in platoon k is as follows:

$$T_{i,k}\,\dot a_{i,k} + a_{i,k} = u_{i,k} \tag{1}$$

where $T_{i,k}, a_{i,k}, u_{i,k}$ are engine's constant, acceleration and control input, respectively.

3. INTERNAL STABILITY ANALYSIS

3. 1. Inter-Platoon Internal Stability

The tracking error between consecutive leaders is defined as:

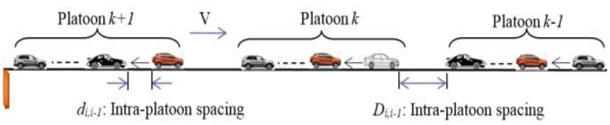


Figure 1. Traffic flow as the inter-connected vehicular platoons

$$\begin{aligned} e_{0,k}(t) &= x_{0,k-1}(t-\tau) - x_{0,k}(t) - \sum\nolimits_{j=0}^{N_{k-1}} L_{j,k-1} - \\ &- \sum\nolimits_{k=1}^{N_{k-1}} h_{k-1}\bar v_{0,k-1} - D_{k,k-1} - h_k\left(v_{0,k} - v^*(\bar e_{0,k})\right) \end{aligned} \tag{2}$$

where x_0, v_0 are position and velocity of lead vehicle, τ is communication delay, L_j is the length of jth vehicle, h_k is time constant headway of kth platoon, N_k is the size of kth platoon, $\bar v_{0,k-1} = \sup\{v_{0,k-1}(t) : t \in [0,\infty)\}$, v^* is a function of inter-platoonspacing which will be defined later and $\bar e_{0,k}$ is defined as follows:

$$\begin{aligned} \bar e_{0,k}(t) &= x_{0,k-1}(t-\tau) - x_{0,k}(t) - \sum\nolimits_{j=0}^{N_{k-1}} L_{j,k-1} - \\ &- \sum\nolimits_{j=1}^{N_{k-1}} h_{k-1}\bar v_{0,k-1} - D_{k,k-1} \end{aligned} \tag{3}$$

To increase the traffic capacity, it is defined that:

$$v^*(\bar e_{0,k}) = \begin{cases} 0, & \bar e_{0,k} < S_1^* \\ \bar v^*(\bar e_{0,k}), & S_1^* \le \bar e_{0,k} \le S_2^* \\ v_{\max}, & \bar e_{0,k} > S_2^* \end{cases} \tag{4}$$

where S_1^*, S_2^* are positive constants and $\bar v^*(\bar e_{0,k})$ is defined as follows:

$$\bar v^*(\bar e_{0,k}) = v_{\max}\left[1 - \cos\left(\pi(\bar e_{0,k} - S_1^*)/(S_2^* - S_1^*)\right)\right]/2 \tag{5}$$

where $v_{\max}$ is the maximum velocity of platoon. By defining new variables $z_{1,k} = e_{0,k}$, $z_{2,k} = \dot e_{0,k}$, $z_{3,k} = \ddot e_{0,k}$ and using Equation (1), the dynamics of each vehicle is as follows:

$$\begin{cases} \dot z_{1,k} = z_{2,k} \\ \dot z_{2,k} = z_{3,k} \\ \dot z_{3,k} = \dot a_{0,k-1}(t-\tau) - u_{0,k}/T_{0,k} + a_{0,k}/T_{0,k} - \\ \quad - h_k \dot u_{0,k}/T_{0,k} + h_k \dot a_{0,k}/T_{0,k} + h_k \ddot v^* \end{cases} \tag{6}$$

In the control design procedure, $\bar u_{0,k}(t)$ is considered as: $\bar u_{0,k}(t) = \bar u_{k,1}(t) + \bar u_{k,2}(t)$. Where $\bar u_{k,1}(t)$ and $\bar u_{k,2}(t)$ satisfy the following expressions:

$$\begin{aligned} &\dot u_{0,k_1}(t) + u_{0,k_1}(t)/h_k = -T_{0,k}u_{0,k}^*(t)/h_k \\ &\dot u_{0,k_2}(t) + u_{0,k_2}(t)/h_k = \dot a_{0,k}(t) + a_{0,k}(t)/h_k + \\ &\quad + \dot a_{0,k-1}(t-\tau)T_{0,k}/h_k + \ddot v^* \end{aligned} \tag{7}$$

By solving Equation (7), $\bar u_{0,k}(t)$ will be in the following form:

$$\begin{aligned} u_{0,k}(t) &= \int_0^t \begin{pmatrix} \dot a_{0,k}(t) + a_{0,k}(t)/h_k + \\ + \dot a_{0,k-1}(t-\tau)T_{0,k}/h_k + \ddot v^* \end{pmatrix} e^{(\alpha - t)/h_k} d\alpha - \\ &- \frac{T_{0,k}}{h_k}\int_0^t u_{0,k}^*(\alpha) e^{(\alpha - t)/h} d\alpha + \left(u_{0,k_1}(0) + u_{0,k_2}(0)\right)e^{-t/h_k} \end{aligned} \tag{8}$$

By defining $u^*_{0,k}(t) = -\alpha z_{1,k}(t) - \beta z_{2,k}(t) - \gamma z_{3,k}(t)$ $\alpha, \beta, \gamma > 0$ and replacing Equation (8) into Equation (6), the closed-loop dynamics of k-th leader is in the following form:

$$\dot{z}_{1,k} = z_{2,k},\ \dot{z}_{2,k} = z_{3,k},\ \dot{z}_{3,k} = \bar{u}^*_{0,k}(t) \quad (9)$$

By applying the parasitic delay, Equation (9) resulted in the following form:

$$\dot{\mathbf{Z}}_k(t) = \mathbf{A}\mathbf{Z}_k(t) + \mathbf{B}\mathbf{Z}_k(t-\Delta_{0,k}),\quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha & -\beta & -\gamma \end{pmatrix} \quad (10)$$

Theorem 1. Under the control law Equation (8) and sufficiently small parasitic delay, the inter-platoon asymptotic stability is assured if:

$$\alpha, \gamma > 0, \quad \gamma\beta > \alpha \quad (11)$$

Proof. The characteristic Equation (10) is as follows:

$$CE_k = \det\left(s\mathbf{I}_2 - \mathbf{A} - \mathbf{B}e^{-\Delta_{0,k}s}\right) = s^3 + \gamma e^{-\Delta_{0,k}s}s^2 + \beta e^{-\Delta_{0,k}s}s + \alpha e^{-\Delta_{0,k}s} = 0 \quad (12)$$

Equation (12) after simplification will be in the following form:

$$-s^3 = (\gamma s^2 + \beta s + \alpha)e^{-\Delta_{0,k}s} \Rightarrow \bar{\varpi}^3 - \gamma^2\bar{\varpi}^2 - (\beta^2 - 2\alpha\gamma)\bar{\varpi} - \alpha^2 = 0, \quad \bar{\varpi} = \omega^2 \quad (13)$$

Since only one sign change occurs in Equation (12), there is only one positive root for $\bar{\varpi}$ for any parametric selection. The phase equality condition of Equation (12) is in the following form.

$$\Delta_{0,k} = \frac{1}{\omega}\left[\tan^{-1}\left(\frac{\beta\omega}{\alpha - \gamma\omega^2}\right) + (4k-1)\frac{\pi}{2}\right] \quad (14)$$

When delay increases from the value calculated by Equation (14), when other parameters are kept fixed, the related imaginary root can cross to the right hand side of *s*-plane. Therefore, the minimum value of $\Delta_{0,k}$ calculated by Equation (14) is the maximum allowable time delay.

3. 2. Intra-platoon Internal Stability Analysis

The intra-platoon tracking error is defined as

$$e_{i,k}(t) = x_{i-1,k}(t) - x_{i,k}(t) - L_{i-1,k} - d_{i-1,k} - \bar{h}_i\left(v_{i,k} - v^*(\bar{e}_{i,k})\right) \quad (15)$$

where $\bar{h}_i$ is intra-platoon time headway and $v^*(\bar{e}_{i,k})$ is defined similar to Equation (4).

Theorem 2. Under the following control law, the intra-platoon asymptotic stability is assured if $\bar{\alpha}, \bar{\gamma} > 0, \bar{\gamma}\bar{\beta} > \bar{\alpha}$

$$u_{i,k}(t) = \int_0^t \begin{pmatrix} \dot{a}_{i,k}(t) + a_{i,k}(t)/h_k + \\ +\dot{a}_{i-1,k}(t)T_{i,k}/\bar{h}_k + \ddot{v}^* \end{pmatrix} e^{(\alpha - t)/\bar{h}_k} d\alpha - \frac{T_{i,k}}{\bar{h}_k}\int_0^t u^*_{i,k}(\alpha)e^{(\alpha-t)/\bar{h}_k}d\alpha + \left(u_{i,k_1}(0) + u_{i,k_2}(0)\right)e^{-t/\bar{h}_k} \quad (16)$$

where, $w_1 = e_{i,k}(t),\ w_2 = \dot{e}_{i,k}(t),\ w_3 = \ddot{e}_{i,k}(t)$ and $u^*_{i,k}(t) = -\bar{\alpha}w_{1,k}(t) - \bar{\beta}w_{2,k}(t) - \bar{\gamma}w_{3,k}(t);\ \bar{\alpha}, \bar{\beta}, \bar{\gamma} > 0.$

Proof. The closed-loop dynamics of each following vehicle is in the following form

$$\dot{\mathbf{W}}(t) = \bar{\mathbf{A}}\mathbf{W}(t) + \bar{\mathbf{B}}\mathbf{W}(t-\Delta_{i,k}),\quad \bar{\mathbf{A}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \bar{\mathbf{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\bar{\alpha} & -\bar{\beta} & -\bar{\gamma} \end{pmatrix} \quad (17)$$

The stability analysis can be completed similar to previous theorem.

4. STRING STABILITY ANALYSIS

4. 1. Inter-platoon String Stability Analysis

Time derivative of both sides of Equation (1), employing Equation (8) and taking Laplace transform of both sides of the resultant equation leads to:

$$G_k(s) = \frac{V_{0,k}(s)}{V_{0,k-1}(s)} = \frac{\left(s^3 + \gamma s^2 + \beta s + \alpha\right)e^{-(\tau + \Delta_{0,k})s}}{h_k s^3 + s^2 + (1-h_k)(\beta s + \alpha)e^{-\Delta_{0,k}s}} \quad (18)$$

Time derivative of Equation (2) and then Laplace transform of it, yields

$$sE_{0,k} = \left[e^{-\tau s} - (1 + h_k s)G_k(s)\right]V_{0,k-1}(s) \quad (19)$$

By performing the same procedure for $e_{0,k-1}(t)$, we have

$$sE_{0,k-1} = \left[e^{-\tau s}/G_{k-1}(s) - (1 + h_{k-1}s)\right]V_{0,k-1}(s) \quad (20)$$

Therefore, the spacing error ratio is as follows:

$$G^*(s) = \frac{E_{0,k}(s)}{E_{0,k-1}(s)} = \frac{\left[e^{-\tau s} - (1+h_k s)G_k(s)\right]G_{k-1}(s)}{e^{-\tau s} - (1 + h_{k-1}s)G_{k-1}(s)} \quad (21)$$

If $\forall\omega : \left|G^*(j\omega)\right| \le 1$, the inter-platoon string stability is assured.

Theorem 3. Under the following condition, the inter-platoon string stability is guaranteed.

$$\alpha > \frac{T_{0,k}}{T^2_{0,k-1}} \quad (22)$$

Proof. $\left|G^*(j\omega)\right| = \sqrt{a/b} \le 1 \Rightarrow b - a \ge 0$. According to Equation (22), it is inferred that:

$$b - a = \sum_{k=0}^{6} \bar{C}_k(\Delta, h, \alpha, \beta)\omega^{2k} \tag{23}$$

The low frequency region is the most determinant region in string stability analysis [1, 4, 7]. Therefore, it is concluded that if $\bar{C}_0 > 0$, the inter-platoon network is string stable. After algebraic manipulations and simplifications, we have $\bar{C}_0 = 4T_{0,k}T_{0,k-1}^2\alpha^3 - T_{0,k}^2\alpha^2$. So that, $\bar{C}_0 > 0$ if Equation (22) holds.

4. 2. Intra-platoon String Stability Analysis

Theorem 4. Under the following condition, then intra-platoon string stability is guaranteed

$$\bar{\alpha} > \frac{T_{0,k}}{T_{0,k-1}^2} \tag{24}$$

Proof. For each vehicle in the platoon we can write

$$G_{i,k}(s) = \frac{V_{i,k}(s)}{V_{i-1,k}(s)} = \frac{\left(s^2 + \bar{\beta}s + \bar{\alpha}\right)e^{-\Delta_{i,k}s}}{\bar{h}_i s^3 + s^2 + (1-\bar{h}_i)(\bar{\beta}s + \bar{\alpha})e^{-\Delta_{i,k}s}}$$

$$\bar{G}^*(s) = \frac{E_{i,k}(s)}{E_{i-1,k}(s)} = \frac{\left[1 - (1 + \bar{h}_i s)G_{i,k}(s)\right]G_{i-1,k}(s)}{1 - (1 + \bar{h}_{i-1}s)G_{i-1,k}(s)}$$

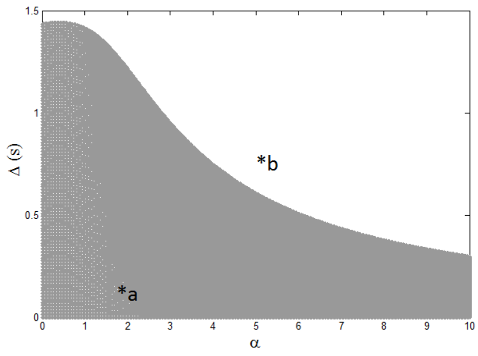
By following the proof of theorem 3, Equation (24) is obtained.

5. SIMULATION STUDIES

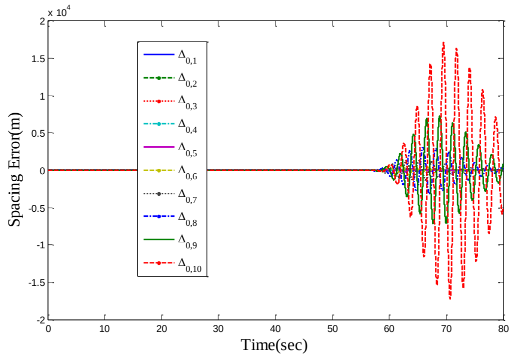
In this section, it is assumed that traffic flow consists of ten heterogeneous vehicular platoons. The inter-platoon and intra-platoon spacing errors are defined as $\delta_{0,k} = x_{0,k-1} - x_{0,k} - \sum_{j=0}^{N_{k-1}} L_{j,k-1} - D_{k,k-1}$ and $\delta_{i,k} = x_{i-1,k} - x_{i,k} - L_{i-1,k-1} - d.$, respectively.

Scenario 1. In this scenario the inter-platoon and intra-platoon stability analyses were studied. Figure 2-a shows the stable region of time delay. In this figure, it is assumed that all control parameters are fixed except α. Therefore, by employing the CTCR method the stable region of delay versus α is presented. Figure 2-b depicts the unstable behavior of platoon for point 'b'. Figure 3 shows the inter-platoon spacing error.

According to this figure, amplitude of error decreases along the platoon indicating the string stability of platoon. Figure 4 shows the velocity of lead vehicles. Figure 5 depicts the spacing error of intra-platoon network. According to this figure, internal and string stability of platoon 3 are assured. Figure 6 shows the velocity of platoon 3.



(a)



(b)

Figure 2. Stable region of delay (a) and unstable behavior (b)

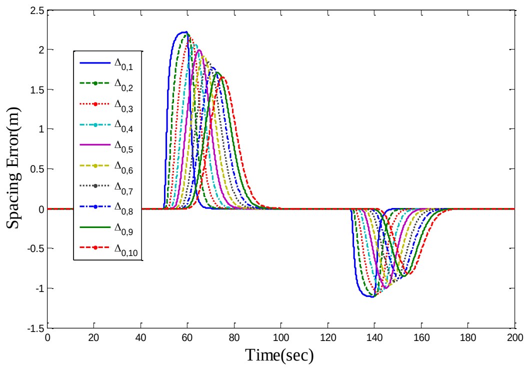


Figure 3. Inter-platoon spacing error

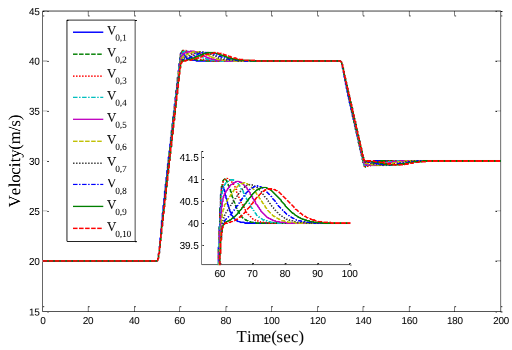


Figure 4. Inter-platoon velocity

Scenario 2: In this scenario, the performance of string stability is studied when a disturbance signal is applied to lead vehicle's motion. Figure 7 shows the performance of string stability of inter-platoon network. Figure 8 illustrates the spacing error of platoon 3.

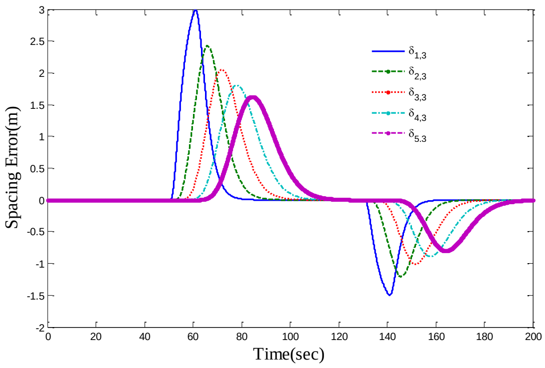


Figure 5. Intra-platoon spacing error

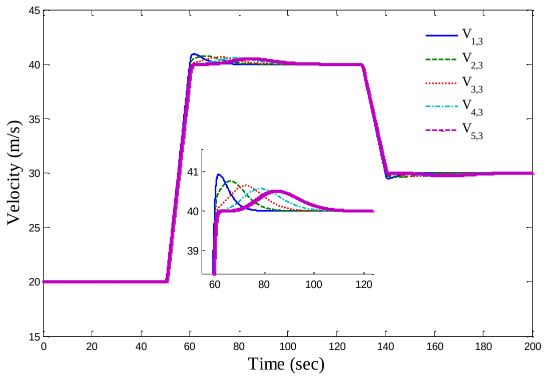


Figure 6. Velocity of platoon 3

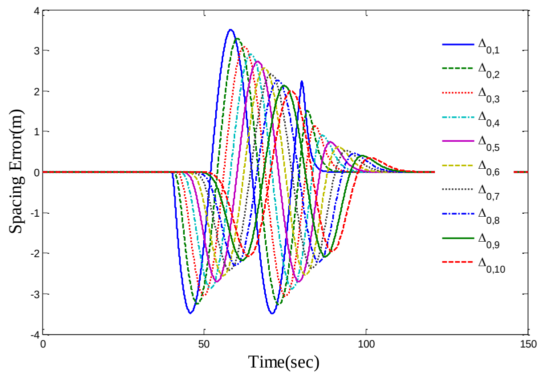


Figure 7. Performance of inter-platoon string stability

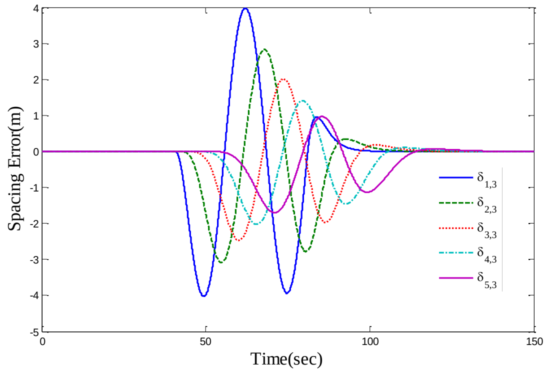


Figure 8. Performance of intra-platoon string stability

6. CONCLUSION

In this paper the problems of internal stability and string stability were studied for heterogeneous traffic flow. It was assumed that the traffic flow consists of infinite number of heterogeneous cooperative vehicular platoons. A new method was introduced to decouple the closed-loop dynamics of inter-platoon and intra-platoon networks. The communication and parasitic delays were investigated in system modeling and control design which is based on CTCR method, the stable region of time delay was introduced. By presenting new theorems, necessary conditions on control parameters assuring asymptotic and string stability for both inter-platoon and intra-platoon networks were presented. Several simulation studies were presented to show the effectiveness of the proposed approaches.

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