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Drift Change Point Estimation in Autocorrelated Poisson Count Processes Using MLE Approach

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ABSTRACT

Change point estimation in the area of statistical process control has received considerable attentions. The assumption of uncorrelated observations is unrealistic in many cases. However, less attention has been given to change point estimation in autocorrelated processes. Among the autocorrelated processes, count data are most widely used in real-world. Different applications of count data are discussed by many researchers such as syndromic surveillance data in healthcare, accident monitoring systems and multi-item pricing models in management science. Poisson distribution for count processes and the first-order integer-valued autoregressive (INAR (1)) model are considered in this paper. We use a combined EWMA and C control chart to monitor the process. We propose change point estimators for the rate and dependence parameters with linear trend under different magnitudes of shifts. For this purpose, Newton's method is used to estimate the paramaters of the process after the change. Then, we develop the maximum likelihood estimators to estimate the real time of change in the parameters. The accuracy and prescision of the proposed MLE estimators are evaluated through simulation studies. In addition, the performance of the proposed estimators is compared with the ones proposed for step change under linear drift. The simulation results confirm that the change point estimators are effective in identifying linear trend in the process parameters. Finally, application of the proposed change point estimators is illustrated through an IP counts data real case.

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1. INTRODUCTION

Statistical process control uses seven tools to reduce variation leading to improvement in the performance of processes. Improving in measurement systems and data storage leads to taking observations very close to each other in time and as a result increasing autocorrelation between observations [1]. Nishina and Wang [2] investigated the performance of cumulative sum (CUSUM) control charts from the view point of the change point estimation considering the autocorrelation. Timmer and Pignatiello [3] develped an MLE approach to determine the change point in the autoregressive parameter, the variance of the white noise and the mean of a first-order autoregressive process. Maximum likelihood change point estimation for the pth-order autoregressive model was proposed by Picard [4]. Perry and Pinatiello [5] extended an MLE to estimate the time of a step change in the mean of stationary and invertible ARMA processes. Perry [6] developed an MLE for the time of polynomial drift in the mean of covariance-stationary autocorrelated processes. Sometimes the process is described by a count data. For instance, the number of customers waiting in a line to be served [7], number of failures in a unit of a product [8] and number of complaints of customers in a service system [9] are examples of count data processes. These cases are usually modeled by the INAR (1) process which is investigated by some researchers such as Al-Osh and Alzaid [10] and Li [11]. Due to several applications of INAR (1) processes in the recent years, this issue is investigated by some researchers. Yontay et al. [12] proposed a Two-sided CUSUM control chart for INAR (1) processes to monitor this process with application to hospital data. Andersson and Karlis [13] evaluated INAR (1) model in the presence of missing data with an

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application to syndromic surveillance data. Yahav and Shmueli [14] proposed a remedial solution for NORTA method to generate Poisson count data with application in management science. Morina et al. [15] used INAR (2) model in order to investigate the number of hospital emergency service arrivals caused by diseases. Also, Weiß [16] proposed confidence regions for both parameters of an INAR (1) model with real application to IP counts data.

Torkamani et al. [17] proposed an MLE approach to estimate the time of a step change in the autocorrelated Poisson count processes modeled by first-order integer-valued autoregressive (INAR(1)). Often deterioration in a machine is shown by a linear change in the parameters. Numerous applications of the autocorrelated count data and Poisson processes were our motivation to investigate this type of the processes. Hence, in this paper we proposed an MLE approach to estimate the time of a linear trend in the autocorrelated Poisson count processes. To monitor the process, we apply the C-EWMA control chart proposed by Weiß [8]. We evaluate the proposed estimator compared to estimator proposed by Torkamani et al. [17] under linear trend disturbance. The results confirm the superiority of the proposed estimator under linear trend. The structure of the paper is as follows: Section 2 provides a description of the INAR (1) model for autocorrelated Poisson count processes. Also, the probability distribution function of the INAR (1) process is introduced. In the next section, maximum likelihood estimators of the change time for the rate and dependence parameters are presented. The performance of the proposed estimators is evaluated in section 4. A real case study related to IP counts data is presented in section 5. Our concluding remarks are given in the final section.

2. THE INAR(1) MODEL AND DISTRIBUTION OF AUTOCORRELATED POISSON COUNT PROCESSES

Common models for stationary real-valued processes are the autoregressive moving average models. However, this model is not usable for integer-valued processes, because multiplication of a real number by an integer value leads to a non-integer value [18]. To overcome this problem, a thinning operation introduced by Steutel and Harn [19] is applied to define integer-valued ARMA models.

Consider X a discrete random variable with range $N_0 = \{0,1,...\}$ and $\alpha \in [0,1]$.

The thinning operation is defined as:

$$\alpha \circ X = \sum_{j=1}^{X} Y_j, \tag{1}$$

where, Y_j is a sequence of independent identically distributed (i.i.d). Bernoulli random variables and independent of X. $\alpha \circ X$ arises from X by binomial thinning and $\circ$ is the binomial thinning operator.

Then, the INAR (1) process was introduced by McKenzie [20], and Al-Osh and Alzaid [10]. The INAR (1) model arises from $M/M/\infty$ queuing system [21].

The recursion of the INAR (1) process is defined by the equation

$$X_t = \alpha \circ X_{t-1} + \varepsilon_t \tag{2}$$

where, X_t is the number of population at a time t, ε_t is the number of new population, $\alpha \circ X_{t-1}$ is the number of population of previous period which are still in the queuing system. For instance, Brannas et al. [22] applied this model for modeling and forecasting guest nights in hotels. The stationary Poisson INAR (1) model assumes that ε_t follows an i.i.d. Poisson (λ) distribution and $X_0 \approx$ Poisson ($\lambda/(1-\alpha)$). Then, X_t is a Markov chain with marginal distribution of Poisson ($\lambda/(1-\alpha)$) [18]. $Y = \alpha \circ X = \sum_{j=1}^{X} Y_j$, follows binomial$(X, \alpha)$.

The probability distribution function of $Z = Y + \varepsilon$ is a convolution of a Poisson and a binomial distribution [23]. Therefore, the probability density function for the INAR (1) process is

$$\begin{aligned} f(x_t \mid x_{t-1}) &= \sum_{k=0}^{m} \binom{x_{t-1}}{k} \alpha^k (1-\alpha)^{x_{t-1}-k} \exp(-\lambda) \frac{\lambda^{x_t - k}}{(x_t - k)!} \\ m &= \min(x_t, x_{t-1}) \end{aligned} \tag{3}$$

Function of an INAR (1) model, derived [24] by multiplication of the probability generating function of all taken samples, should be written based on its conditional distribution as:

$$f_{x_0,\dots,x_T}(x_0,\dots,x_T \mid \alpha,\lambda) = f_{x_0}(x_0) \cdot \prod_{t=1}^{T} f_{x_t \mid x_{t-1}}(x_t \mid x_{t-1}) \tag{4}$$

The probability density function of first observation is defined as follows:

$$f_{x_0}(x_0) = \exp[-(\lambda/(1-\alpha))](\lambda/(1-\alpha))^{x_0} / x_0! \tag{5}$$

3. PROPOSED CHANGE POINT ESTIMATOR UNDER LINEAR TREND

In this section, we derive a maximum likelihood estimator to find the real time of the drift change point in the rate (λ) and the dependence (α) parameters, where we consider a linear trend model for the mentioned parameters.

3. 1 Change-point Estimator for the Rate Parameter It is assumed that an INAR (1) process initially is in-control status with known rate parameters

$\lambda = \lambda_0$. After an unknown point in time a change in the process occurs and the value of $\lambda = \lambda_0$ changes to an unknown value where $\lambda_{1i} = \lambda_0 + \beta_1(i-\tau)$ for $i = \tau+1,...,T$ and T is the time when a control chart signals an out-of-control status.

We propose a MLE approach to estimate the change point when a linear trend occurs in the rate parameter of an autocorrelated Poisson process. The log-likelihood function for this process can be written as:

$$\ell(\lambda_{1i},\tau \mid x) = \frac{-\lambda_0}{(1-\alpha)} + \ln\left[\frac{(\lambda_0/(1-\alpha))^{x_0}}{x_0!}\right] + \sum_{i=1}^{\tau}\ln(x_{t-1}!) - \tau(\lambda_0) + \sum_{i=1}^{\tau}\ln[\sum_{k=0}^{m}\frac{\alpha^k(1-\alpha)^{x_{j,t-1}-k}\lambda_0^{x_{j,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!}] + \sum_{i=\tau+1}^{T}\ln(x_{t-1}!) - \sum_{i=\tau+1}^{T}(\lambda_0+\beta_1(i-\tau)) + \sum_{i=\tau+1}^{T}\ln[\sum_{k=0}^{m}\frac{\alpha^k(1-\alpha)^{x_{j,t-1}-k}(\lambda_0+\beta_1(i-\tau))^{x_{j,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!}] \quad (6)$$

Because the slope parameter β_1 is unknown, by taking the partial derivative of Equation (6) with respect to β_1, we obtain

$$\frac{\partial\ell(\lambda_1,\tau \mid x)}{\partial\beta_1} = \{-\frac{1}{2}(T-\tau)(T+1-\tau) + \sum_{i=\tau+1}^{T}[\frac{\sum_{k=0}^{m}\frac{\alpha^k(1-\alpha)^{x_{t-1}-k}(i-\tau)(\lambda_0+\beta_1(i-\tau)))^{x_{t-k-1}}}{k!(x_t-k-1)!(x_{t-1}-k)!}}{\sum_{k=0}^{m}\frac{\alpha^k(1-\alpha)^{x_{t-1}-k}(\lambda_0+\beta_1(i-\tau)))^{x_{t-k}}}{k!(x_t-k)!(x_{t-1}-k)!}}]\} \quad (7)$$

As it is obvious in the above equation, there is no closed-form solution for the slope parameter (β_1). Hence, to overcome this problem, we apply Newton's method to approximate the slope parameter. For more details; see [25] and [26]. Newton's method is a derivative based algorithm that uses the linear approximation in order to search and find the root of an equation in the given model.

$$f'(x+\Delta x) \cong f'(x) + \frac{d}{dx}f'(x)\Delta x \quad (8)$$

where $\Delta x = x_{k+1} - x_k$. If set $f'(x+\Delta x)$ equal to zero, the following equation is obtained:

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}. \quad (9)$$

Note that the initial point x_0 and an appropriate stopping scheme lead the algorithm to converge at the root of f'. Since τ is unknown, for each potential τ the algorithm is repeated and the value of $\hat{\beta}_{1\tau}$ is obtained through the following equation:

$$\hat{\beta}_{1\tau,k+1} = \hat{\beta}_{1\tau,k} - \frac{f'(x)}{f''(x)} \quad (10)$$

The computations of $f'(x)$ and $f''(x)$ are given in Appendix A. Finally, substituting $\hat{\beta}_1$ for β_1 in Equation (11), we obtain the estimated change point as follows:

$$\hat{\tau} = \arg\max\{\sum_{i=1}^{\tau}\ln(x_{t-1}!) - \tau(\lambda_0) + \sum_{i=\tau+1}^{T}\ln(x_{t-1}!) + \sum_{i=1}^{\tau}\ln[\sum_{k=0}^{m}\frac{\alpha^k(1-\alpha)^{x_{i,t-1}}\lambda_0^{x_{i,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!} +] \sum_{i=\tau+1}^{T}(\lambda_0+\hat{\beta}_1(i-\tau)) + \sum_{i=\tau+1}^{T}\ln[\sum_{k=0}^{m}\frac{\alpha^k(1-\alpha)^{x_{i,t-1}-k}(\lambda_0+\hat{\beta}_1(i-\tau))^{x_{i,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!}]\} \quad (11)$$

3. 2. Change-point Estimator for the Dependence Parameter

It is assumed that an INAR (1) process starts with value of the known dependence parameter $\alpha = \alpha_0$ for the first τ observations. A change in the process occurs and the value of $\alpha = \alpha_0$ changes to an unknown value where $\alpha_{1i} = \alpha_0 + \beta_2(i-\tau)$ for $i = \tau+1,...,T$ where T is the time when a control chart signals an out-of-control state. Note that the dependence paramater is between zero and one. Derivation of the log-likelihood function for the mentioned process is as follows:

$$\ell(\alpha_{1i},\tau \mid x) = \frac{-\lambda}{(1-\alpha_0)} + \ln\left[\frac{(\lambda/(1-\alpha_0))^{x_0}}{x_0!}\right] + \sum_{i=1}^{\tau}\ln(x_{t-1}!) - \tau(\lambda) + \sum_{i=1}^{\tau}\ln(\sum_{k=0}^{m}\frac{\alpha_0^{k}(1-\alpha_0)^{x_{j,t-1}-k}\lambda^{x_{j,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!}) + \sum_{i=\tau+1}^{T}\ln(x_{t-1}!) - \lambda(T-\tau) + \sum_{i=\tau+1}^{T}\ln[\sum_{k=0}^{m}\frac{(\alpha_0+\beta_2(i-\tau))^k(1-(\alpha_0+\beta_2(i-\tau)))^{x_{j,t-1}-k}(\lambda)^{x_{j,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!}] \quad (12)$$

The slope parameter β_2 is unknown. Hence, the partial derivative of Equation (12) with respect to β_2 leads to

$$\frac{\sum_{k=0}^{m}k(i-\tau)(\alpha_0+\beta_2(i-\tau))^{k-1}(1-(\alpha_0+\beta_2(i-\tau)))^{x_{i,t-1}}}{\sum_{k=0}^{m}(\alpha+\beta_2(i-\tau))^k} \times \frac{-(\alpha_0+\beta_2(i-\tau))^k(i-\tau)(x_{i,t-1}-k)}{(1-(\alpha_0+\beta_2(i-\tau)))^{x_{i,t-1}-k}} \times \frac{(1-(\alpha_0+\beta_2(i-\tau)))^{x_{i,t-1}}\ /k!(x_{i,t-1}-k)!(x_{i,t}-k)!}{\lambda^{x_{j,t}-k}\ /k!(x_{j,t}-k)!(x_{j,t-1}-k)!} \quad (13)$$

In Equation (13), there is no closed-form solution for the slope parameter (β_2). To approximate for the slope parameter, we use Newton's method according to Equation (14) and estimate $\hat{\beta}_{2\tau}$ as:

$$\hat{\beta}_{2\tau,k+1} = \hat{\beta}_{2\tau,k} - \frac{f'(x)}{f''(x)} \quad (14)$$

See Appendix B for calculations of $f'(x)$ and $f''(x)$.

We obtained the change point estimate of τ by Substituting $\hat{\beta}_2$ in the following equation:

$$\hat{\tau}=\operatorname{argmax}\{\sum_{i=1}^{\tau}\ln(x_{t-1}!)+\sum_{i=\tau+1}^{T}\ln(x_{t-1}!)+ \sum_{i=1}^{\tau}\ln[\sum_{k=0}^{m}\frac{(\alpha_0+\hat{\beta}_2(i-\tau))^k(1-(\alpha_0+\hat{\beta}_2(i-\tau)))^{x_{i,t-1}-k}\lambda^{x_{i,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!}]+ \sum_{i=\tau+1}^{T}\sum_{i=\tau+1}^{T}\ln[\sum_{k=0}^{m}\frac{(\alpha_0+\hat{\beta}_2(i-\tau))^k(1-(\alpha_0+\hat{\beta}_2(i-\tau)))^{x_{i,t-1}-k}\lambda^{x_{i,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!}]\} \quad (15)$$

3. 3. Monitoring Scheme Weiß [10] proposed a control chart based on a combination of the C and a EWMA control chart to monitor the INAR (1) process. Due to the suitable performance of the combined C-EWMA charts, it is used in this paper to monitor the rate (λ) and dependent (α) parameters. The EWMA statistic is defined as

$$Q_t = round(\lambda_{EWMA}\cdot X_t + (1-\lambda_{EWMA})\cdot Q_{t-1}) \quad (16)$$

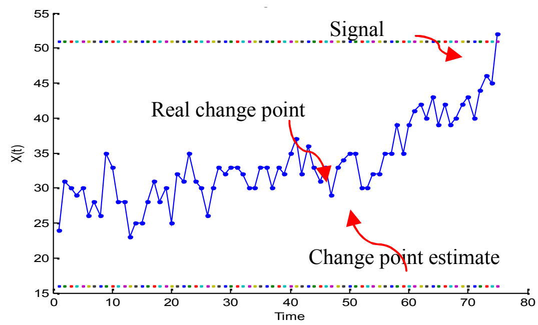
where, $Q_0 = round(\lambda/(1-\alpha))$ and $\lambda_{EWMA}\in(0,1]$ is the smoothing parameter. We considered (X_t,Q_t) as statistics that are plotted simultaneously on a C-chart and a EWMA chart.

4. PERFORMANCES OF THE PROPOSED CHANGE POINT ESTIMATORS

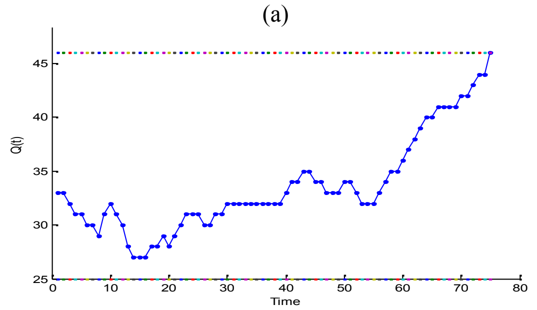
In this section, the performance of the proposed estimators for rate and dependence parameters is investigated by a Monte Carlo simulation when a linear trend change is occurred. MATLAB software[2] is used for comparison study through 5000 simulation runs. At first, we indicate a diagnostic plot to illustrate the use of the change point estimator for the rate and the dependence parameters. Then, we use a Monte Carlo simulation to make performance comparisons between the proposed estimatorsand received signal from the C-EWMA control charts. Also, we consider a comparison study between estimator of linear trend and step change when a linear trend occurs in the process.

4. 1. Evaluating the Change Point Estimator of the Rate Parameter We present a diagnostic plot shown in Figure 1. The time is indicated by horizontal axis and vertical axis represents the value of statistic $X(t)$. We assume there is an INAR (1) process and generate 50 observations with $\lambda_0=10$ and $\alpha=0.7$. Then, the rate parameter changes to an out-of-control status as $\lambda_{1i}=\lambda_0+\beta_1(i-\tau)$ where the slope parameter is equal to $\beta_1=0.2$. The control chart signals at observation 75. As shown in Figure 1, the maximum log-likelihood value is obtained at t=54. The result shows that the change point estimator performs well in estimating the real time of the change and estimates the change point closer to real change point respect to the signal time. To evaluate the performance of the proposed estimators, we consider an INAR (1) processes with parameters $\alpha=0.1$,0.4 and 0.7, $\lambda=6$,4 and 10. Table 1 shows the control limits used for monitoring autocorrelated Poisson count processes proposed by Weiß [8]. The averages of change point estimator for the linear trend and step change under different values of dependence parameter and various shifts of the rate parameter are indicated in Figure 2. As shown in this figure when the shift size is small, the C-EWMA control chart indicates a poor performance to distinguish the signal. Furthermore, when the value of the slope parameter increases, the estimator will provide a more accurate and precise estimate of τ. Also, we obtain that when a linear trend disturbance is occurred, $\hat{\tau}_{lt}$ is more accurate than the $\hat{\tau}_{sc}$.

The precision of the proposed estimators is investigated by standard deviation and mean square errors (MSE), when a linear trend disturbance occurs in the process. The results in Table 2 show that in most cases under linear trend disturbance, the mean square errors of the linear trend estimator are less than the ones for the step change point estimator. Also, the accuracy of the two change point estimators significantly improves by increasing the value of the slope parameter.



(a)



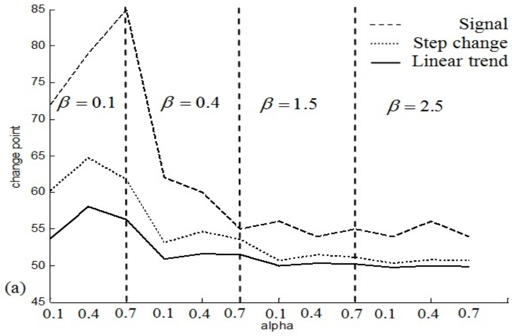
(b)

Figure 1. (a) C control chart (b) EWMA control chart and the change point estimator of the rate parameter $\alpha=0.7$, $\beta_1=0.2$, $\lambda_0=10$

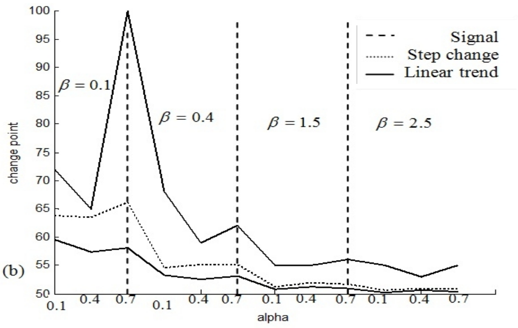
[2]MATLAB and Statistics Toolbox Release 2009b, The MathWorks, Inc.

TABLE 1. Control limits of the C and EWMA control charts (Weiß[8])

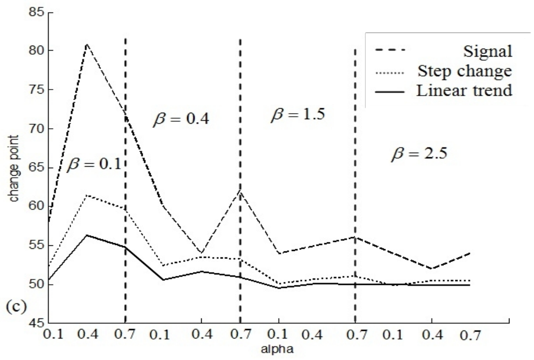
α	λ EWMA	lc	uc	le	ue
			$\lambda=10$		
0.1	0.1	2	21	9	14
0.4	0.1	5	29	14	21
0.7	0.05	18	50	30	39
			$\lambda=6$		
0.1	0.75	0	15	4	13
0.4	0.1	1	19	8	14
0.7	0.1	8	34	14	25
			$\lambda=4$		
0.1	0.25	0	11	3	8
0.4	0.75	0	15	1	13
0.7	0.1	4	24	10	18



(a)



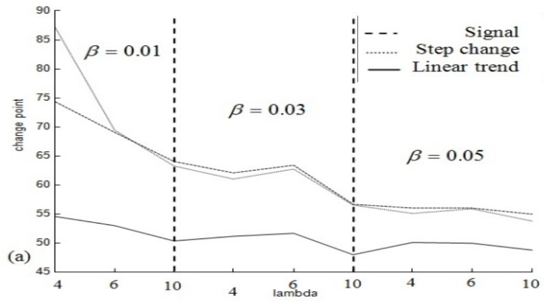
(b)



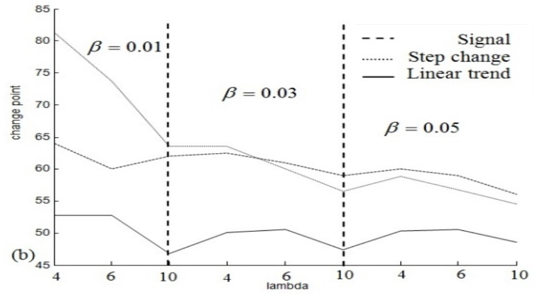
(c)

Figure 2. Average change point estimates for the rate parameter (a) $\lambda=10$, (b) $\lambda=6$, and (c) $\lambda=4$

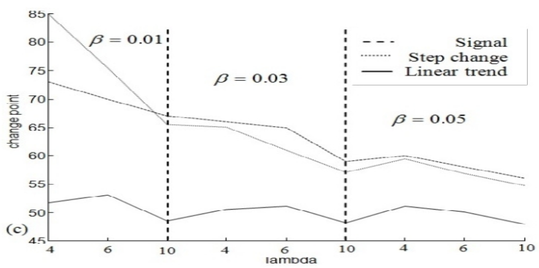
4. 2. Evaluating the Change Point Estimator of the Dependence Parameter We generate observations from the INAR (1) process with parameters $\alpha = 0.1$, 0.4 and 0.7, and $\lambda=10$, 6 and 4. To monitor the process, the C-EWMA control chart is used. As shown in Table 1, the control limits proposed by Weiß [8] is used for monitoring autocorrelated Poisson count processes. The results of simulation runs for the average of change point estimates for the linear trend and step change under different values of dependence parameter and various shifts of the rate parameter are illustrated in Figure 3. We conclude from this figure that under small magnitudes of shift, the performance of the C-EWMA control chart in detecting signal decreases significantly. However, by increasing the value of slope parameter, the proposed estimator estimates the real change point more accurately and precisely. Also, the $\hat{\tau}_{lt}$ outperforms the $\hat{\tau}_{sc}$ under different linear trend disturbances.



(a)



(b)



(c)

Figure 3. Average of change point estimates for the dependence parameter (a) $\alpha=0.1$, (b) $\alpha=0.4$ and , (c) $\alpha=0.7$

TABLE 2. Precision criteria including standard error and MSE of the proposed rate parameter estimator in comparison with the step estimator with $\alpha = 0.1, 0.4, 0.7$ under different shifts

α		$\lambda=10$			
	β_1	0.1	0.4	1.5	2.5
0.1	se($\bar{\bar{\tau}}_{lt}$)	0.250	0.11	0.06	0.05
	se($\bar{\bar{\tau}}_{sc}$)	0.197	0.10	0.06	0.05
	MSE($\bar{\bar{\tau}}_{lt}$)	342.6	65.6	17.1	10.8
	MSE($\bar{\bar{\tau}}_{sc}$)	347.9	61.6	17.2	11.6
0.4	se($\bar{\bar{\tau}}_{lt}$)	0.274	0.155	0.069	0.054
	se($\bar{\bar{\tau}}_{sc}$)	0.214	0.123	0.068	0.056
	MSE($\bar{\bar{\tau}}_{lt}$)	123.1	53.02	11.05	7.177
	MSE($\bar{\bar{\tau}}_{sc}$)	150.2	48.11	12.28	8.222
0.7	se($\bar{\bar{\tau}}_{lt}$)	0.245	0.11	0.05	0.03
	se($\bar{\bar{\tau}}_{sc}$)	0.191	0.09	0.04	0.03
	MSE($\hat{\tau}_{lt}$)	305.8	66.4	13.4	9.5
	MSE($\bar{\bar{\tau}}_{sc}$)	407.3	61.5	11.5	8.8
α		$\lambda=6$			
0.1	se($\bar{\bar{\tau}}_{lt}$)	0.23	0.12	0.07	0.05
	se($\bar{\bar{\tau}}_{sc}$)	0.18	0.10	0.07	0.06
	MSE($\bar{\bar{\tau}}_{lt}$)	243.6	63.9	20.9	12.7
	MSE($\bar{\bar{\tau}}_{sc}$)	244.5	57.6	20.1	17.4
0.4	se($\bar{\bar{\tau}}_{lt}$)	0.235	0.111	0.067	0.044
	se($\bar{\bar{\tau}}_{sc}$)	0.179	0.090	0.05	0.048
	MSE($\bar{\bar{\tau}}_{lt}$)	28.90	8.76	2.62	1.459
	MSE($\bar{\bar{\tau}}_{sc}$)	35.72	9.37	2.14	1.838
0.7	se($\bar{\bar{\tau}}_{lt}$)	0.212	0.11	0.06	0.050
	se($\bar{\bar{\tau}}_{sc}$)	0.168	0.09	0.05	0.043
	MSE($\bar{\bar{\tau}}_{lt}$)	219.6	53.7	10.3	10.1
	MSE($\bar{\bar{\tau}}_{sc}$)	252.9	51.2	12.1	7.9
α		$\lambda=4$			
0.1	se($\bar{\bar{\tau}}_{lt}$)	0.1243	0.1223	0.0808	0.0366
	se($\bar{\bar{\tau}}_{sc}$)	0.1225	0.1137	0.0890	0.0829
	MSE($\bar{\bar{\tau}}_{lt}$)	11.960	12.745	4.1117	2.8653
	MSE($\bar{\bar{\tau}}_{sc}$)	12.869	12.658	4.911	5.5490
0.4	se($\bar{\bar{\tau}}_{lt}$)	0.2233	0.1049	0.050	0.0493
	se($\bar{\bar{\tau}}_{sc}$)	0.1610	0.0939	0.061	0.0514
	MSE($\bar{\bar{\tau}}_{lt}$)	38.517	11.999	1.3523	1.3367
	MSE($\bar{\bar{\tau}}_{sc}$)	42.253	12.879	2.1303	1.5950
0.7	se($\bar{\bar{\tau}}_{lt}$)	0.2362	0.1318	0.061	0.0511
	se($\bar{\bar{\tau}}_{sc}$)	0.1950	0.1015	0.0526	0.0510
	MSE($\bar{\bar{\tau}}_{lt}$)	33.409	10.766	6.8707	4.1113
	MSE($\bar{\bar{\tau}}_{sc}$)	36.527	8.461	5.7630	4.1923

TABLE 3. Precision criteria including standard error and MSE of the proposed rate parameter estimator in comparison with the step estimator with $\alpha = 0.1$, $\alpha = 0.4$ and 0.7 under different shifts

α		$\lambda=10$		
	β_2	0.01	0.03	0.05
0.1	se($\hat{\tau}_{lt}$)	0.2517	0.1622	0.1291
	se($\hat{\tau}_{sc}$)	0.1338	0.0518	0.0396
	MSE ($\hat{\tau}_{lt}$)	11.71	8.13	3.4830
	MSE ($\hat{\tau}_{sc}$)	36.00	13.83	3.8800
0.4	se($\hat{\tau}_{lt}$)	0.1652	0.0921	0.0737
	se($\hat{\tau}_{sc}$)	0.0674	0.0285	0.0197
	MSE ($\hat{\tau}_{lt}$)	139.095	42.461	27.911
	MSE ($\hat{\tau}_{sc}$)	395.81	80.099	36.045
0.7	se($\hat{\tau}_{lt}$)	0.1336	0.0896	0.0722
	se($\hat{\tau}_{sc}$)	0.0373	0.0168	0.0125
	MSE ($\hat{\tau}_{lt}$)	92.167	43.680	31.086
	MSE($\hat{\tau}_{sc}$)	161.831	32.114	15.034
		$\lambda=6$		
0.1	se($\hat{\tau}_{lt}$)	0.3143	0.2084	0.1789
	se($\hat{\tau}_{sc}$)	0.1841	0.0706	0.0443
	MSE ($\hat{\tau}_{lt}$)	304.204	130.27	96.103
	MSE ($\hat{\tau}_{sc}$)	1080	199.64	83.537
0.4	se($\hat{\tau}_{lt}$)	0.2649	0.1245	0.0696
	se($\hat{\tau}_{sc}$)	0.1148	0.0448	0.0304
	MSE ($\hat{\tau}_{lt}$)	35.45	5.562	2.879
	MSE ($\hat{\tau}_{sc}$)	108.6	12.808	9.596
0.7	se($\hat{\tau}_{lt}$)	0.1922	0.1726	0.1536
	se($\hat{\tau}_{sc}$)	0.0487	0.0322	0.0248
	MSE ($\hat{\tau}_{lt}$)	158.84	65.8315	32.49
	MSE($\hat{\tau}_{sc}$)	189.17	44.1650	20.67
		$\lambda=4$		
0.1	se($\hat{\tau}_{lt}$)	0.360	0.238	0.1683
	se($\hat{\tau}_{sc}$)	0.188	0.068	0.0461
	MSE ($\hat{\tau}_{lt}$)	98.534	29.341	7.625
	MSE ($\hat{\tau}_{sc}$)	333.901	41.444	8.413
0.4	se($\hat{\tau}_{lt}$)	0.249	0.138	0.121
	se($\hat{\tau}_{sc}$)	0.138	0.051	0.035
	MSE ($\hat{\tau}_{lt}$)	49.622	114.673	13.624
	MSE($\hat{\tau}_{sc}$)	178.097	31.92	15.956
0.7	se($\hat{\tau}_{lt}$)	0.266	0.153	0.143
	se($\hat{\tau}_{sc}$)	0.066	0.029	0.023
	MSE ($\hat{\tau}_{lt}$)	76.867	51.028	21.508
	MSE($\hat{\tau}_{sc}$)	125.838	36.709	7.858

Finally, we use Monte Carlo simulation to evaluate the precision of the proposed estimators through their standard deviation and their mean square errors (MSE), when a linear trend change occurs in the process. The results are summarized in Table 3. We can conclude from this table that the values of mean square errors related to linear trend change point estimator are less than the MSEs of the step change point estimator in most cases. Also, the accuracy of two change point estimators significantly improves by increasing the value of the slope parameters.

Table 4 indicates percentage of the time, the difference between the real and the estimated change points falls within various intervals. Note that the first and second rows in each cell show the precision of the proposed and step estimators, respectively. The results show that as the magnitude of the linear shift in the process mean increases, the percentage of falling the change point estimate in the specific interval from the real change point decreases. It shows the proposed estimator has more acceptable precision rather than the step estimator. The same results are obtained for the rate parameter and other values of dependence parameter (not reported here).

5. A REAL CASE: IP COUNTS DATA

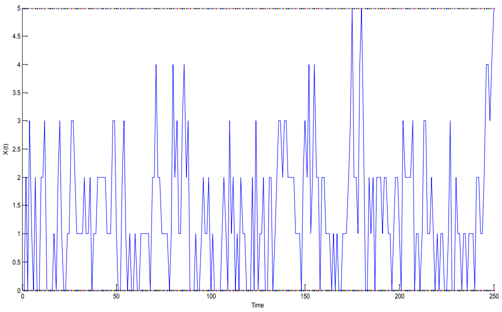
We use a real IP counts data that collected by the server of the statistic Department University of Wurzburg [9]. This data collected on time interval of November and December 2005, between 10 o'clock in the morning and 6 o'clock in the evening with 240 time series observations. In this case, count data is defined as the number of different users or IP addresses access the server within period of length 2 min. If an INAR (1) model is fitted on the data set, the parameters of the model are estimated equal to $\alpha = 0.29$ and $\lambda = 0.91$ [11]. We adjusted ALR_0 equal to 273.67 by setting the control limits of C and EWMA control charts equal to *lc*=0, *uc*=5, *le*=1, and *ue*=4 through 10000 simulation runs.

Finally, the proposed change point estimators are used to find the real time of linear trend changes in the parameters of the INAR(1) process. A linear trend change occurs in the process at observation 241 with the slope parameter $\beta_1 = 0.3$. EWMA control chart signals at 250th observation. The maximum log-likelihood value is obtained at *t*=244 by using the rata parameter change point estimator. Also, the step change point estimator find the real time of a change at *t*=248.

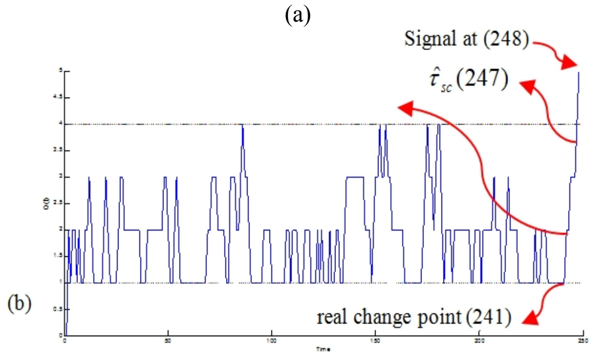
We also evaluate the performance of the change point estimator under linear trend in dependence parameter. A linear trend occurs in the process with the slope parameter $\beta_2 = 0.05$. As illustrated in Figure 5, the EWMA control chart signals at 248th observation. The maximum log-likelihood value is obtained at *t*=242. This value for the step change estimator is obtained at *t*=247. The results show that the proposed estimators estimate the real change points accurately and it confirm that the model is applicable for real world applications as well.

TABLE 4. Precision of the change point estimator for the dependent parameter

α	$\lambda = 10$			
	β_2 / Δu	0.01	0.03	0.05
0.1	$p\lvert\hat{\tau}-\tau\rvert = 0$	0.0023 (0.0003)	0.0083 (0.0003)	0.0063 0.0007)
	$p\lvert\hat{\tau}-\tau\rvert \le 1$	0.005 (0.0004)	0.0183 (0.0007)	0.0163 (0.0017)
	$p\lvert\hat{\tau}-\tau\rvert \le 2$	0.009 0.0007	0.0317 (0.001)	0.0260 (0.0027)
	$p\lvert\hat{\tau}-\tau\rvert \le 3$	0.0127 0.0003	0.0423 (0.001)	0.0340 (0.0043)
	$p\lvert\hat{\tau}-\tau\rvert \le 5$	0.0173 0.0007	0.0573 (0.0033)	0.0520 (0.0137)
	$p\lvert\hat{\tau}-\tau\rvert \le 6$	0.0203 0.0013	0.0643 (0.005)	0.0553 (0.0240)
0.4	$p\lvert\hat{\tau}-\tau\rvert = 0$	0.046 (0.0001)	0.097 (0.0003)	0.1468 (0.003)
	$p\lvert\hat{\tau}-\tau\rvert \le 1$	0.131 (0.0003)	0.283 (0.0007)	0.399 (0.009)
	$p\lvert\hat{\tau}-\tau\rvert \le 2$	0.220 (0.002)	0.436 (0.0023)	0.582 (0.015)
	$p\lvert\hat{\tau}-\tau\rvert \le 3$	0.301 (0.002)	0.573 (0.0013)	0.740 (0.0157)
	$p\lvert\hat{\tau}-\tau\rvert \le 5$	0.444 (0.0033)	0.773 (0.0033)	0.911 (0.031)
	$p\lvert\hat{\tau}-\tau\rvert \le 6$	0.508 (0.0047)	0.841 (0.0063)	0.936 (0.089)
0.7	$p\lvert\hat{\tau}-\tau\rvert = 0$	0.081 (0.0001)	0.129 (0.0003)	0.052 (0.0001)
	$p\lvert\hat{\tau}-\tau\rvert \le 1$	0.247 (0.0001)	0.335 (0.0007)	0.197 (0.00013)
	$p\lvert\hat{\tau}-\tau\rvert \le 2$	0.390 (0.001)	0.499 (0.0033)	0.434 (0.0103)
	$p\lvert\hat{\tau}-\tau\rvert \le 3$	0.505 (0.0011)	0.632 (0.01)	0.683 (0.066)
	$p\lvert\hat{\tau}-\tau\rvert \le 5$	0.673 (0.0013)	0.821 (0.083)	0.914 (0.184)
	$p\lvert\hat{\tau}-\tau\rvert \le 6$	0.732 (0.0037)	0.883 (0.148)	0.937 (0.188)

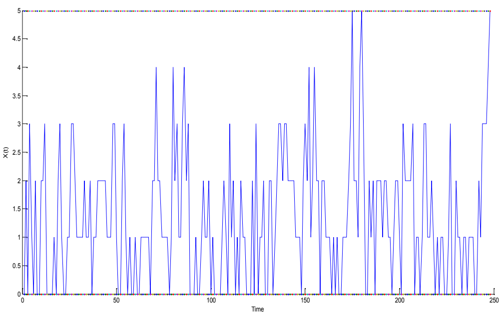


(a)

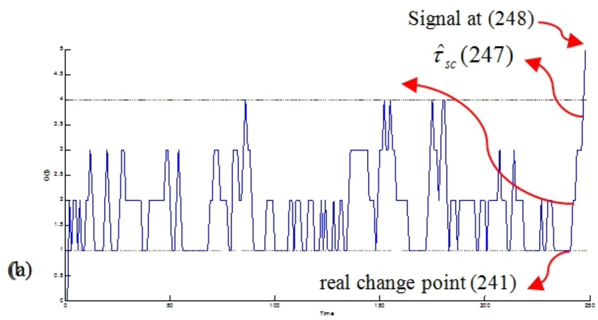


(b)

Figure 4. (a) C control chart (b) EWMA control chart and the change point estimator for the rate parameter for IP counts data



(a)



(b)

Figure 5. (a) C control chart (b) EWMA control chart and the change point estimator for the dependence parameters for IP counts data

6. CONCLUDING REMARKS

In this paper, a first-order integer-valued autoregressive (INAR (1)) model was considered. We compared the proposed estimators of linear trend with the corresponding step change point estimators when a linear trend occurs in the process. The results confirmed the superiority of the proposed estimators to estimate the real time of linear trend changes in the process parameters rather than the step change point estimators. We also investigated the effect of autocorrelation coefficient, the smoothing parameter of the EWMA control chart and the rate parameter of the Poisson distribution on the accuracy and precision of change point estimators. The results showed that the proposed estimators outperform the corresponding step change point estimators under all situations. As future researches, we suggest developing change point estimators for autocorrelated geometric, and binomial distributions under different shifts. Also, investigating the effect of missing data on the performance change point estimators developed in this research as well as the other researches in the area of change point estimation such as Keramatpour et al. [27] and Fallahnezhad et al. [28] and proposing some remedial methods based on Ashuri and Amiri [29] can be a fruitful area for future research.

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APPENDIX A

DERIVATIONS RELATED TO NEWTONS METHOD FOR THE RATE PARAMETER

For the rate parameter, we assume

$$U=\sum_{K=0}^{m}\frac{\alpha^{k}(1-\alpha)^{x_{j,t-1}-k}(\lambda_0+\beta_1(i-\tau))^{x_{j,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!} \qquad (A_1)$$

then, the first derivative of the log likelihood function with respect to the rate parameter is calculated as

$$\frac{\partial\ell(\lambda_1,\tau\mid x)}{\partial\beta_1}=\{-\frac{1}{2}(T-\tau)(T+1-\tau)+\sum_{i=\tau+1}^{T}[\frac{U'_{\beta_1}}{U}]\}. \qquad (A_2)$$

Also, the second derivative of the log likelihood function is equal to

$$\frac{\partial^2\ell(\lambda_1,\tau\mid x)}{\partial\beta_1^2}=\frac{U''_{\beta_1}U-{U'_{\beta_1}}^2}{U^2} \qquad (A_3)$$

Where

$$U'_{\beta_1}=\sum_{k=0}^{m}\frac{{\alpha_0}^{k}(1-\alpha_0)^{x_{t-1}-k}(i-\tau)(\lambda_0+\beta_1(i-\tau))^{x_t-k-1}}{k!(x_t-k-1)!(x_{t-1}-k)!} \qquad (A_4)$$

and

$$U''_{\beta_1}=\sum_{k=0}^{m}\frac{\alpha_0(1-\alpha_0)^{x_{t-1}-k}(i-\tau)^2(\lambda_0+\beta_1(i-\tau))^{x_t-k-2}}{k!(x_t-k-2)!(x_{t-1}-k)!}. \qquad (A_5)$$

APPENDIX B

DERIVATIONS RELATED TO NEWTONS METHOD FOR THE DEPENDENCE PARAMETER

The appendix presents computations of $f'(x), f''(x)$ in order to estimate the slope parameter in the newton's method.
For the dependence parameter, we assume

$$V=\sum_{K=0}^{m}\frac{(\alpha_0+\beta_2(i-\tau))^{k}(1-(\alpha_0+\beta_2(i-\tau)))^{x_{j,t-1}-k}\lambda^{x_{j,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!} \qquad (B_1)$$

The first derivative of thelog likelihood function is calculated as follows:

$$\frac{\partial \ell(\alpha_1,\tau \mid x)}{\partial_{\beta_2}} = \sum_{t=\tau+1}^{T} \frac{V'_{a_1}}{V}, \quad (B_2)$$

where

$$V'_{\beta_2} = \sum_{i=0}^{m} \frac{\lambda^{x_{j,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!}[k(i-\tau)(a_0+\beta_2(i-\tau))^{k-1}$$
$$(1-(\alpha_0+\beta_2(i-\tau)))^{x_{j,t-1}}(\alpha_0+\beta_2(i-\tau))^{k}(i-\tau)$$
$$(x_{j,t-1}-k)(1-(\alpha_0+\beta_2(i-\tau)))^{x_{j,t-1}-k-1}] \quad (B_3)$$

Also, the second derivative of the log likelihood function is

$$\frac{\partial^2 \ell(\alpha_1,\tau \mid x)}{\partial \beta_2^2} = \frac{V''_{\beta_2}V - V'^2_{B_2}}{V^2} \quad (B_4)$$

Where

$$V'_{\beta_2} = \sum_{K=0}^{m} \frac{\lambda^{x_{j,t}-k}}{k!(x_{j,t}-k)!(x_{j,t-1}-k)!}[k(i-\tau)^2$$
$$\times(\alpha_0+\beta_2(i-\tau))^{k-2}(k-1)(1-(\alpha_0+\beta_2(i-\tau)))^{x_{j,t-1}-k}$$
$$\times k(i-\tau)^2(\alpha_0+\beta_2(i-\tau))^{k-1}(x_{j,t-1}-k)$$
$$\times(1-(\alpha_0+\beta_2(i-\tau)))^{x_{j,t-1}-k-1}-(\alpha_0+\beta_2(i-\tau))^{k-1}$$
$$(i-\tau)^2k(x_{j,t-1}-k)(1-(\alpha_0+\beta_2(i-\tau)))^{x_{j,t-1}-k-1}$$
$$+(\alpha_0+\beta_2(i-\tau))(i-\tau)^2(x_{j,t-1}-k)$$
$$\times(x_{j,t-1}-k-1)(1-(\alpha_0+\beta_2(i-\tau)))^{x_{j,t-1}-k-2} \quad (B_5)$$

Drift Change Point Estimation in Autocorrelated Poisson Count Processes Using MLE Approach

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چکیده

تخمین نقطه تغییر در حوزه کنترل فرایند آماری مورد توجه زیادی قرار گرفته است. در بسیاری از کاربرد های دنیای واقعی فرض استقلال مشاهدات غیر واقعی است. با این وجود تخمین نقطه تغییر در فرآیندهای خود همبسته کمتر مورد توجه قرار گرفته است. در بین فرآیند های خودهمبسته، داده های شمارشی بیشترین کاربرد را در دنیای واقعی دارند. داده های نظارت بر سندروم در حوزه بهداشت و درمان، سیستم های پایش تصادف و مدل های قیمت گذاری چند منظوره در علم مدیریت نمونه هایی از کاربرد های داده های شمارشی است که مورد بررسی و بحث توسط محققان قرار گرفته است. توزیع پواسان به همراه مدل اتورگرسیون عدد صحیح مرتبه اول دراین مقاله در نظر گرفته شده است. به منظور پایش فرآیند از نمودار کنترل C- EWMA استفاده شده است. تخمین زننده نقطه تغییر برای پارامترهای نرخ وقوع و وابستگی تحت شیفت تدریجی ارائه شده است. روش نیوتن به منظور تخمین پارامترهای فرآیند بعد از نقطه تغییر استفاده شده است. سپس تخمین زننده های ماکزیمم درستنمایی به منظور تخمین زمان واقعی تغییر در پارامتر ها توسعه داده شده است. صحت و دقت تخمین زننده ارائه شده از طریق شبیه سازی مورد ارزیابی قرار گرفته است. بعلاوه، عملکرد تخمین زنندههای تحت شیفت تدریجی با تخمین زننده های پله ای مورد مقایسه قرار گرفته است. نتایج شبیه سازی کارآیی براورد کننده پیشنهادی را در کشف شیفت های تدریجی تایید می کند. در پایان کاربرد مدل پیشنهادی برای داده های واقعی (داده های شمارشی پروتکل اینترنت) نشان داده شده است.

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