

# AN ELASTICITY SOLUTION FOR STATIC ANALYSIS OF FUNCTIONALLY GRADED CURVED BEAM SUBJECTED TO A SHEAR FORCE

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**Abstract** In this paper, using 2-D theory of elasticity, a closed-form solution is presented for stress distributions and displacements of a FG curved beam under shear force at its free end. The material properties are assumed to vary continuously through the radial direction based on a simple power law model and Poisson's ratio is supposed to be constant. In order to verify the solution, it is shown that all stress and displacement relations are converted to those of a homogenous curved beam when the inhomogeneity constant approaches zero. The effects of inhomogeneity on stress distributions are investigated. It is shown that specified stress distribution profiles can be obtained by changing the variation of volume fraction of constituents. It is observed that for a specific value of inhomogeneity constant, a proper stress distribution along the radial direction is obtained for designing purposes.

**Keywords** Two-Dimensional Elasticity, Closed-Form Solution, Functionally Graded Material, Curved Beam

**چکیده** در این مقاله با استفاده از تئوری الاستیسیته دو بعدی برای توزیع مولفه‌های تنش و جابجایی یک تیر ساخته شده از مواد هدفمند FG تحت بار برشی در لبه آزاد خود حل پاسخ-بسته ارائه شده است. فرض شده است که خواص مواد سازنده تیر بر اساس یک مدل تابع توانی ساده به طور پیوسته در راستای شعاعی تغییر می‌کند و نسبت پواسون ثابت در نظر گرفته شده است. به منظور بررسی صحت حل ارائه شده، نشان داده شده است که با میل کردن ثابت غیرهمگنی به سمت صفر، همه روابط تنش و جابجایی به روابط مربوط به تیر همگن تبدیل می‌شود. همچنین، تاثیرات ناهمگنی بر روی توزیع مولفه‌های تنش بررسی شده است. نشان داده شده است که با تغییر در توزیع کسر حجمی مواد سازنده تیر می‌توان به توزیع تنش مطلوب و مشخصی رسید. نتایج حاصل از حل نشان می‌دهد که به ازای مقدار مشخصی از ثابت غیرهمگنی، توزیع تنش مطلوبی در راستای شعاعی حاصل می‌گردد که برای مقاصد طراحی مناسب است.

## 1. INTRODUCTION

Planar circular arches, one of the basic structural components, are omnipresent in most common engineering design structures such as arched bridges, turbo-machinery blades, aircraft and rail structures, etc. Therefore, the understanding of stress distributions and displacements of such curved beams is one of the most important issues for designing these structures.

In recent years, the development of a new class of materials known as "functionally graded materials" (FGMs) in which the material properties vary continuously in one or more directions according to a specific profile attracted interests for designing and manufacturing of solid structures. They were first introduced in 1984 by material scientists in Japan [1]. Due to the smooth variation of material properties, they offer many advantages over conventional materials.

The material properties within a functionally graded circular plane bar vary in the radial direction, but are constant across the depth. This type of inhomogeneity can be due to several causes: directional cooling leading to a micro-structural gradient, phase segregation arising as a result of centrifugal casting, property degradation of the fuel cladding in nuclear reactors, chemical and vapor deposition, and surface modification using laser technology [2].

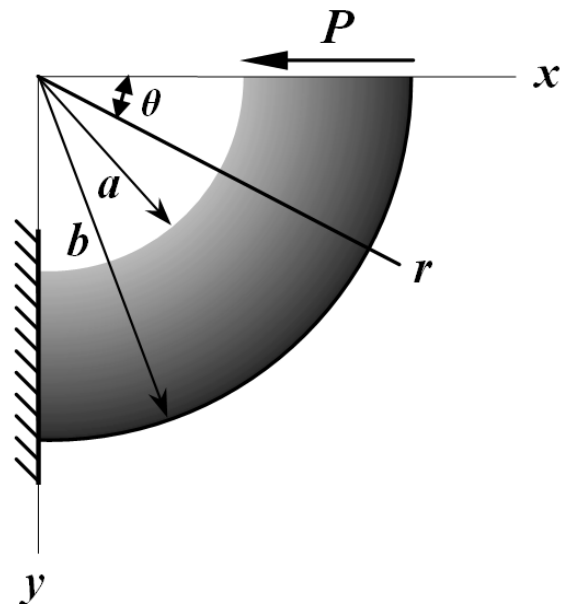
There exist some studies in literature related to analyses of planar curved beams both made of isotropic and FGMs. Yu and Nie [3] presented an analytical solution for the shearing and radial stresses in curved beams based on the solution for a Volterra integral equation of the second kind. Their proposed formulae satisfy both the equilibrium equations and the static boundary conditions on the surfaces of the beams. Shafiee et al. [4] studied in-plane and out-of-plane buckling of functionally graded arches. Lim et al. [5] investigated temperature-dependent in-plane vibration of functionally graded circular arches based on the two-dimensional theory of elasticity. They obtained an analytical solution for a simply supported circular arch using the state space formulation and Fourier series. Malekzadeh [6] studied the in-plane free vibration analysis of functionally graded thick circular arches subjected to initial stresses due to thermal environment based on two-dimensional theory of elasticity. Later, Malekzadeh et al. [7] analyzed the in-plane free vibration of functionally graded thin-to-moderately thick deep circular arches in thermal environments based on first-order shear deformation theory (FSDT). Saidi et al. [8,9] presented exact solutions for stresses and displacements of functionally graded sector plates and spheres. Jomehzadeh et al. [10] investigated static analysis of thick FG sector plates according to a new analytical approach. Hosseini-Hashemi et al. [11] studied free vibration and buckling of FG sector plates on elastic foundation using differential quadrature method. An analytical solution for free vibration of curved and twisted beam was presented by Yu et al [12].

The companion problem of finding the stress distributions and displacements of circular arches and curved beams has not attracted much attention and there are only a few studies in the literature related to this issue. Dryen [2] considered the

axisymmetric problem of stress distribution across a functionally graded circular beam subjected to pure bending. However, no works were found in the literature for static analysis of the non-axisymmetric problem of a functionally graded cantilever curved beam subjected to a shear force at its end. Hence, the present study is carried out to provide a closed-form solution for the problem based on the two-dimensional theory of elasticity. By using the present solution, the effects of inhomogeneity are investigated to obtain a proper stress distribution across a FG curved beam for designing purposes. It is shown that the obtained relations for stress and displacement components are converted to those of homogeneous beam when the inhomogeneity is vanished.

## 2. PROBLEM ANALYSIS

Consider a FG curved beam of a narrow rectangular cross section, which is clamped at the lower end and shear force  $P$  is applied at the upper end in the radial direction (Fig. 1). The material properties are assumed to vary continuously through the radial direction, according to the volume fraction of the constituents.



**Figure 1.** Geometry and coordinate system of a FG curved beam subjected to shear force at its upper end.

The two-dimensional theory of elasticity in polar coordinates is applied here to analyze the problem. To start the analysis, the compatibility equation in polar coordinates is utilized, as follows

$$\frac{\partial}{\partial r} \left( r \frac{\partial \gamma_{r\theta}}{\partial \theta} - r^2 \frac{\partial \varepsilon_\theta}{\partial r} \right) + r \frac{\partial \varepsilon_r}{\partial r} - \frac{\partial^2 \varepsilon_\theta}{\partial \theta^2} = 0 \quad (1)$$

Considering plane stress state, the stress-strain relations for isotropic FG curved beam are defined as

$$\begin{aligned} \varepsilon_r &= \frac{1}{E(r)} (\sigma_r - \nu \sigma_\theta) \\ \varepsilon_\theta &= \frac{1}{E(r)} (\sigma_\theta - \nu \sigma_r) \\ \gamma_{r\theta} &= \frac{2(1+\nu)}{E(r)} \tau_{r\theta} \end{aligned} \quad (2)$$

where  $E(r)$  is Young's modulus and considered to vary through the radial direction only and  $\nu$  is Poisson's ratio and is assumed to be constant. It is noticeable that in analyses of functionally graded materials, the Poisson's ratio is usually held constant (e.g. see [13-15]).

Substituting Eq. (2) into Eq. (1), the compatibility equation in terms of stress components for functionally graded materials is obtained as

$$\begin{aligned} &\left( \nu r^2 \frac{\partial^2}{\partial r^2} + (2\nu + 1)r \frac{\partial}{\partial r} - \frac{\partial^2}{\partial \theta^2} \right) \sigma_r - r^2 \frac{E''(r)}{E(r)} (\nu \sigma_r - \sigma_\theta) \\ &- \left( r^2 \frac{\partial^2}{\partial r^2} + (\nu + 2)r \frac{\partial}{\partial r} - \nu \frac{\partial^2}{\partial \theta^2} \right) \sigma_\theta \\ &+ 2(\nu + 1) \left( \frac{\partial}{\partial \theta} + r \frac{\partial^2}{\partial r \partial \theta} \right) \tau_{r\theta} + 2r^2 \left( \frac{E'(r)}{E(r)} \right)^2 (\nu \sigma_r - \sigma_\theta) \\ &- r \frac{E'(r)}{E(r)} \left[ \left( 2\nu r \frac{\partial}{\partial r} + (2\nu + 1) \right) \sigma_r - \left( 2r \frac{\partial}{\partial r} + (2 + \nu) \right) \sigma_\theta \right. \\ &\left. + 2(\nu + 1) \frac{\partial}{\partial \theta} \tau_{r\theta} \right] = 0 \end{aligned} \quad (3)$$

In addition to the above equation, equilibrium equations in polar coordinates should also be solved. These equations in the absence of body forces are written as follows

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad (4.1)$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0 \quad (4.2)$$

It is assumed that the material properties of FG curved beam follow a simple power-law distribution as [16]

$$\eta(r) = \eta_0 r^\beta \quad (5)$$

where  $\eta$  is one of the effective material properties such as Young's modulus and  $\beta$  is the inhomogeneity constant.

### 3. SOLUTION OF THE GOVERNING EQUATIONS

Using the separation of variables approach and trigonometric functions, the in-plane components of stress field are considered as

$$\begin{aligned} \sigma_r &= f(r) \sin \theta \\ \sigma_\theta &= g(r) \sin \theta \\ \tau_{r\theta} &= h(r) \cos \theta \end{aligned} \quad (6)$$

where  $f$ ,  $g$  and  $h$  are unknown functions of variable  $r$ . The proposed stress field (6) satisfies both the compatibility equation and the equilibrium equations. Substituting Eq. (6) into Eqs. (3), (4.1) and (4.2), a set of ordinary differential equation is obtained as

$$\begin{aligned} &\nu r^2 f''(r) + [(2\nu + 1) - 2\nu\beta] r f'(r) \\ &+ [1 - (\nu + 1)\beta + \nu\beta^2] f(r) - r^2 g''(r) \\ &- [(\nu + 2) - 2\beta] r g'(r) - [\nu - (\nu + 1)\beta + \beta^2] g(r) \\ &- 2(\nu + 1) r h'(r) - 2(\nu + 1)(1 - \beta) h(r) = 0 \end{aligned} \quad (7.1)$$

$$r f'(r) + f(r) - g(r) - h(r) = 0 \quad (7.2)$$

$$r h'(r) + g(r) + 2h(r) = 0 \quad (7.3)$$

Solving the above set of equations and simplifying the results yields

$$\begin{Bmatrix} f(r) \\ g(r) \\ h(r) \end{Bmatrix} = \begin{bmatrix} -\alpha_1 & \alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 1 & 1 & 1 \\ -1 & -\alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix} \begin{Bmatrix} C_1 r^{-1} \\ C_2 r^A \\ C_3 r^B \\ C_4 r^X \end{Bmatrix} \quad (8)$$

in which  $C_1$  through  $C_4$  are constants which are obtained from the boundary conditions. The coefficients  $\alpha_1$  through  $\alpha_4$  are also defined as

$$\begin{aligned}\alpha_1 &= \frac{3 + \nu + \beta}{1 - \nu(1 + \beta)} \\ \alpha_2 &= \frac{1}{1 + \beta} \\ \left\{ \begin{matrix} \alpha_3 \\ \alpha_4 \end{matrix} \right\} &= \frac{2 + \beta \mp \sqrt{16 + 4\beta(-\nu) + \beta^2}}{2(3 - \nu\beta)}\end{aligned}\quad (9)$$

Furthermore,  $A$ ,  $B$  and  $X$  are as follows

$$\begin{aligned}A &= -1 + \beta \\ \left\{ \begin{matrix} B \\ X \end{matrix} \right\} &= \frac{-2 + \beta \pm \sqrt{16 + 4\beta(-\nu) + \beta^2}}{2}\end{aligned}\quad (10)$$

Taking a look at Eq. (8) reveals that setting the constant  $C_1$  to be zero yields a relation between the functions  $f(r)$  and  $h(r)$  as

$$f(r) + h(r) = 0 \quad (11)$$

It is noted that the existence of Eq. (11) is necessary for satisfying all the boundary conditions of the problem. Therefore, the in-plane components of stress field can be expressed as

$$\sigma_r = (\alpha_2 C_2 r^A - \alpha_3 C_3 r^B - \alpha_4 C_4 r^X) \sin\theta \quad (12.1)$$

$$\sigma_\theta = (C_2 r^A + C_3 r^B + C_4 r^X) \sin\theta \quad (12.2)$$

$$\tau_{r\theta} = (-\alpha_2 C_2 r^A + \alpha_3 C_3 r^B + \alpha_4 C_4 r^X) \cos\theta \quad (12.3)$$

Now, one can consider the zero normal and shear stress boundary conditions at the inner and outer boundaries of the curved beam as follows

$$\sigma_r|_{r=a} = \sigma_r|_{r=b} = \tau_{r\theta}|_{r=a} = \tau_{r\theta}|_{r=b} = 0 \quad (13)$$

The above boundary conditions will result in the following two independent equations

$$\begin{cases} \alpha_2 a^A C_2 - \alpha_3 a^B C_3 = \alpha_4 a^X C_4 \\ \alpha_2 b^A C_2 - \alpha_3 b^B C_3 = \alpha_4 b^X C_4 \end{cases} \quad (14)$$

By solving Eqs. (14), the constants  $C_2$  and  $C_3$  are obtained in term of  $C_4$  as

$$\begin{aligned}C_2 &= \left( \frac{\alpha_4}{\alpha_2} \right) \left( \frac{a^X b^B - a^B b^X}{a^A b^B - a^B b^A} \right) C_4 \\ C_3 &= \left( \frac{\alpha_4}{\alpha_3} \right) \left( \frac{a^X b^A - a^A b^X}{a^A b^B - a^B b^A} \right) C_4\end{aligned}\quad (15)$$

According to Fig. 1, the sum of the shearing forces distributed over the upper end of the curved beam should be equal to the shear force  $P$ . Assuming a FG curved beam of a narrow rectangular cross section and taking the width of the cross section as unity, this boundary condition is written as follows

$$\int_a^b \tau_{r\theta} \Big|_{\theta=0} dr = P \quad (16)$$

Substituting Eqs. (15) and (12.3) into Eq. (16), an algebraic equation in term of  $C_4$  is obtained. By solving this equation, the constant  $C_4$  is determined as

$$\begin{aligned}C_4 &= - \left( \frac{P}{\alpha_4} \right) \left[ \left( \frac{a^X b^B - a^B b^X}{a^A b^B - a^B b^A} \right) \left( \frac{b^{A+1} - a^{A+1}}{A+1} \right) \right. \\ &\quad \left. - \left( \frac{a^X b^A - a^A b^X}{a^A b^B - a^B b^A} \right) \left( \frac{b^{B+1} - a^{B+1}}{B+1} \right) - \left( \frac{b^{X+1} - a^{X+1}}{X+1} \right) \right]^{-1}\end{aligned}\quad (17)$$

Substituting Eq. (17) into Eqs. (15) and (12), the stress components are obtained as

$$\begin{aligned}\sigma_r &= \Delta \left[ - (a^X b^B - a^B b^X) r^A + (a^X b^A - a^A b^X) r^B \right. \\ &\quad \left. + (a^A b^B - a^B b^A) r^X \right] P \sin\theta \\ \sigma_\theta &= \Delta \left[ -\alpha_2^{-1} (a^X b^B - a^B b^X) r^A - \alpha_3^{-1} (a^X b^A - a^A b^X) r^B \right. \\ &\quad \left. - \alpha_4^{-1} (a^A b^B - a^B b^A) r^X \right] P \sin\theta \\ \tau_{r\theta} &= \Delta \left[ (a^X b^B - a^B b^X) r^A - (a^X b^A - a^A b^X) r^B \right. \\ &\quad \left. - (a^A b^B - a^B b^A) r^X \right] P \cos\theta\end{aligned}\quad (18.1)$$

where

$$\begin{aligned}\Delta &= \left[ (a^X b^B - a^B b^X) \left( \frac{b^{A+1} - a^{A+1}}{A+1} \right) \right. \\ &\quad - (a^X b^A - a^A b^X) \left( \frac{b^{B+1} - a^{B+1}}{B+1} \right) \\ &\quad \left. - (a^A b^B - a^B b^A) \left( \frac{b^{X+1} - a^{X+1}}{X+1} \right) \right]^{-1}\end{aligned}\quad (18.2)$$

Now, we consider the displacements produced by the shear force P. To this end, using Eqs. (2) and (12), the strain components are defined as

$$\varepsilon_r = \frac{1}{E_0} [(\alpha_2 - \nu)C_2 r^{-1} - (\alpha_3 + \nu)C_3 r^{B-\beta} - (\alpha_4 + \nu)C_4 r^{X-\beta}] \sin\theta \quad (19.1)$$

$$\varepsilon_\theta = \frac{1}{E_0} [(1 - \nu\alpha_2)C_2 r^{-1} + (1 + \nu\alpha_3)C_3 r^{B-\beta} + (1 + \nu\alpha_4)C_4 r^{X-\beta}] \sin\theta \quad (19.2)$$

$$\gamma_{r\theta} = \frac{1}{E_0} [-2(1 + \nu)\alpha_2 C_2 r^{-1} + 2(1 + \nu)\alpha_3 C_3 r^{B-\beta} + 2(1 + \nu)\alpha_4 C_4 r^{X-\beta}] \cos\theta \quad (19.3)$$

The strain-displacement relations in polar coordinates are expressed as

$$\varepsilon_r = \frac{\partial u}{\partial r} \quad (20.1)$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \quad (20.2)$$

$$\gamma_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \quad (20.3)$$

Substituting the radial strain component from Eq. (19.1) into Eq. (20.1) and integrating the results with respect to r, the radial displacement is obtained as follows

$$u = \frac{1}{E_0} \left[ (\alpha_2 - \nu)C_2 \ln r - \frac{(\alpha_3 + \nu)C_3}{B - \beta + 1} r^{B-\beta+1} - \frac{(\alpha_4 + \nu)C_4}{X - \beta + 1} r^{X-\beta+1} \right] \sin\theta + f_1(\theta) \quad (21)$$

in which  $f_1$  is a function of  $\theta$  only. Substituting  $\varepsilon_\theta$  from Eq. (19.2) and also the radial displacement, u, from Eq. (21) into Eq. (20.2), and integrating the results with respect to  $\theta$ , the following relation is obtained for circumferential displacement

$$v = \frac{1}{E_0} \left[ - \left( 1 + \nu\alpha_3 + \frac{\alpha_3 + \nu}{B - \beta + 1} \right) C_3 r^{B-\beta+1} - (1 - \nu\alpha_2)C_2 + (\alpha_2 - \nu)C_2 \ln r - \left( 1 + \nu\alpha_4 + \frac{\alpha_4 + \nu}{X - \beta + 1} \right) C_4 r^{X-\beta+1} \right] \cos\theta - \int f_1(\theta) d\theta + g_1(r) \quad (22)$$

where  $g_1$  is only a function of r. Substituting Eqs. (21), (22) and (19.3) into (20.3), a differential equation is obtained as follows

$$\int f_1(\theta) d\theta + f_1'(\theta) + r g_1'(r) - g_1(r) = - \frac{C_2}{E_0} [(1 - \nu) + (3 + \nu)\alpha_2] \cos\theta \quad (23)$$

Solving the above equation, the unknown functions  $f_1(\theta)$  and  $g_1(r)$  are obtained as

$$f_1(\theta) = - \frac{C_2}{2E_0} [(1 - \nu) + (3 + \nu)\alpha_2] \theta \cos\theta + \ell_1 \sin\theta + \ell_2 \cos\theta$$

$$g_1(r) = \ell_3 r \quad (24)$$

where  $\ell_1$ ,  $\ell_2$  and  $\ell_3$  are three constants and can be determined by satisfying the clamped edge conditions of the curved beam at  $\theta = \pi/2$ . These conditions are expressed as

$$v|_{\theta=\pi/2} = 0 \quad (25.1)$$

$$u|_{\theta=\pi/2} = 0 \quad (25.2)$$

$$\partial u / \partial \theta|_{\theta=\pi/2} = 0 \quad (25.3)$$

Using Eqs. (21), (22) and (24), the Eqs. (25) can be rewritten as follows

$$\frac{C_2}{2E_0} [(1 - \nu) + (3 + \nu)\alpha_2] \frac{\pi}{2} - \ell_2 + \ell_3 r = 0 \quad (26.1)$$

$$\frac{1}{E_0} \left[ (\alpha_2 - \nu)C_2 \ln r - \frac{(\alpha_3 + \nu)C_3}{B - \beta + 1} r^{B-\beta+1} - \frac{(\alpha_4 + \nu)C_4}{X - \beta + 1} r^{X-\beta+1} \right] + \ell_1 = 0 \quad (26.2)$$

$$\frac{C_2}{2E_0} [(1 - \nu) + (3 + \nu)\alpha_2] \frac{\pi}{2} - \ell_2 = 0 \quad (26.3)$$

By solving Eqs. (26.1) and (26.2), the constants  $\ell_2$  and  $\ell_3$  are obtained,

$$\ell_2 = \frac{\pi C_2}{4 E_0} [(1 - \nu) + (3 + \nu)\alpha_2] \quad (27)$$

$$\ell_3 = 0$$

Applying the condition (26.1) for the middle

surface of the beam, yields

$$\ell_1 = -\frac{1}{E_0} \left[ (\alpha_2 - \nu) C_2 \ln \left( \frac{a+b}{2} \right) - \frac{(\alpha_3 + \nu) C_3 \left( \frac{a+b}{2} \right)^{B-\beta+1}}{B-\beta+1} - \frac{(\alpha_4 + \nu) C_4 \left( \frac{a+b}{2} \right)^{X-\beta+1}}{X-\beta+1} \right] \quad (28)$$

Thus, the deflection of the upper end of the FG curved beam is determined as

$$u|_{\theta=0} = -\frac{\pi[(1-\nu) + (3+\nu)\alpha_2](a^X b^B - a^B b^X)}{4E_0 \alpha_2} \times \left[ \frac{(a^X b^B - a^B b^X)(b^{A+1} - a^{A+1})}{A+1} - \frac{(a^X b^A - a^A b^X)(b^{B+1} - a^{B+1})}{B+1} - \frac{(a^A b^B - a^B b^A)(b^{X+1} - a^{X+1})}{X+1} \right]^{-1} P \quad (29)$$

By using the above relation, the effect of inhomogeneity on the beam deflection is considerable.

#### 4. VERIFICATION OF THE SOLUTION

The developed elasticity solution for a functionally graded curved beam can be validated by comparison with the available results in the literature for isotropic homogeneous curved beam. Apparently, stresses and displacements in the homogeneous curved beam can be obtained from the proposed solution for a FG curved beam by numerically setting  $\beta \rightarrow 0$  in the corresponding relations. In this case, the coefficients  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  and also A, B and X are simplified as follows

$$\alpha_1 = \frac{3+\nu}{1-\nu}, \quad \alpha_2 = 1, \quad \alpha_3 = -\frac{1}{3}, \quad \alpha_4 = 1 \quad (30)$$

A = -1, B = 1, X = -3

The only indeterminate term in Eq. (17) is  $(b^{A+1} - a^{A+1})/(A+1)$ . The limit of this term when the inhomogeneity constant approaches zero is calculated as

$$\lim_{\beta \rightarrow 0} \frac{b^{A+1} - a^{A+1}}{A+1} = \ln \left( \frac{b}{a} \right) \quad (31)$$

Using Eqs. (30) and (31), the stress components (18) are simplified for homogenous curved beam as follows

$$\sigma_r = \frac{\left( r + \frac{a^2 b^2}{r^3} - \frac{1}{r}(a^2 + b^2) \right) P \sin \theta}{a^2 - b^2 + (a^2 + b^2) \ln \left( \frac{b}{a} \right)}$$

$$\sigma_\theta = \frac{\left( 3r - \frac{a^2 b^2}{r^3} - \frac{1}{r}(a^2 + b^2) \right) P \sin \theta}{a^2 - b^2 + (a^2 + b^2) \ln \left( \frac{b}{a} \right)} \quad (32)$$

$$\tau_{r\theta} = -\frac{\left( r + \frac{a^2 b^2}{r^3} - \frac{1}{r}(a^2 + b^2) \right) P \cos \theta}{a^2 - b^2 + (a^2 + b^2) \ln \left( \frac{b}{a} \right)}$$

The above relations are the same as relations in Ref. [17] for homogenous curved beam. Also, the limit of Eq. (29) when the inhomogeneity constant approaches zero is determined

$$u|_{\theta=0} = -\frac{P\pi(a^2 + b^2)}{E_0 \left[ (a^2 - b^2) + (b^2 + a^2) \ln \left( \frac{b}{a} \right) \right]} \quad (33)$$

Equation (33) is exactly the same as the relation in Ref. [17] for the deflection of the upper end of a homogenous curved beam.

#### 5. NUMERICAL RESULTS AND DISCUSSION

Based on the analysis of the foregoing sections, the stresses and displacements of a cantilever FG curved beam subjected to shear force are obtained. Numerical results are presented for various values of the inhomogeneity constant,  $\beta$ , and various inner-to-outer radius ratios.

Table 1 shows the maximum values of non-dimensional stresses and their location in curved beams. It is seen that the maximum values of normal and shear stresses occur at the built-in end ( $\theta = \pi/2$ ) and free end ( $\theta = 0$ ) of the beam, respectively. Also, it is concluded that the variation of the inhomogeneity constant affects the amount of maximum stresses and their radial location in

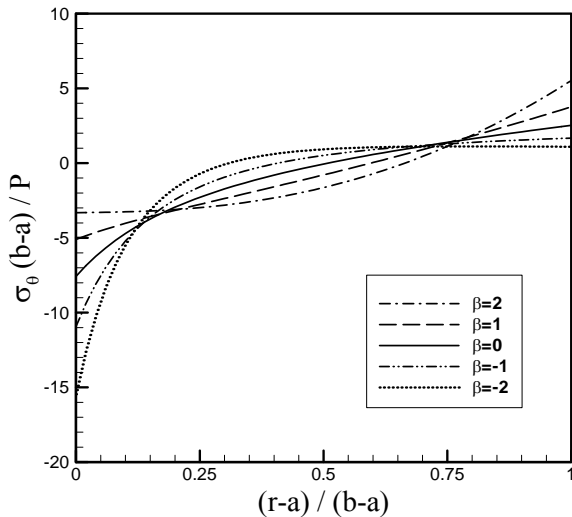
**TABLE 1. Maximum values of non-dimensional stresses and their location in curved beams**

$a/b$	$\beta$	$Location (\bar{r}, \theta)$	$\bar{\sigma}_r$	$Location (\bar{r}, \theta)$	$\bar{\sigma}_\theta$	$Location (\bar{r}, \theta)$	$\bar{\tau}_{r,\theta}$
1/8	2	(0.4857, $\pi/2$ )	-1.4313	(1, $\pi/2$ )	+6.6837	(0.4857, 0)	1.4313
	1	(0.2097, $\pi/2$ )	-1.5560	(0, $\pi/2$ )	-5.1513	(0.2097, 0)	1.5560
	0	(0.0975, $\pi/2$ )	-2.2807	(0, $\pi/2$ )	-12.221	(0.0975, 0)	2.2807
	-1	(0.0646, $\pi/2$ )	-3.4095	(0, $\pi/2$ )	-24.985	(0.0646, 0)	3.4095
	-2	(0.0480, $\pi/2$ )	-4.8677	(0, $\pi/2$ )	-45.742	(0.0480, 0)	4.8677
1/4	2	(0.4482, $\pi/2$ )	-1.4427	(1, $\pi/2$ )	+7.1141	(0.4482, 0)	1.4427
	1	(0.2886, $\pi/2$ )	-1.5345	(0, $\pi/2$ )	-6.2529	(0.2886, 0)	1.5345
	0	(0.1901, $\pi/2$ )	-1.7942	(0, $\pi/2$ )	-10.506	(0.1901, 0)	1.7942
	-1	(0.1382, $\pi/2$ )	-2.1766	(0, $\pi/2$ )	-16.765	(0.1382, 0)	2.1766
	-2	(0.1069, $\pi/2$ )	-2.6657	(0, $\pi/2$ )	-25.782	(0.1069, 0)	2.6657
1/2	2	(0.4510, $\pi/2$ )	-1.4827	(1, $\pi/2$ )	+10.403	(0.4510, 0)	1.4827
	1	(0.3873, $\pi/2$ )	-1.5109	(0, $\pi/2$ )	-10.159	(0.3873, 0)	1.5109
	0	(0.3312, $\pi/2$ )	-1.5656	(0, $\pi/2$ )	-12.882	(0.3312, 0)	1.5656
	-1	(0.2840, $\pi/2$ )	-1.6445	(0, $\pi/2$ )	-16.194	(0.2840, 0)	1.6445
	-2	(0.2451, $\pi/2$ )	-1.7459	(0, $\pi/2$ )	-20.224	(0.2451, 0)	1.7459
3/4	2	(0.4767, $\pi/2$ )	-1.4970	(1, $\pi/2$ )	+22.112	(0.4767, 0)	1.4970
	1	(0.4523, $\pi/2$ )	-1.5020	(0, $\pi/2$ )	-22.073	(0.4523, 0)	1.5020
	0	(0.4284, $\pi/2$ )	-1.5109	(0, $\pi/2$ )	-24.295	(0.4284, 0)	1.5109
	-1	(0.4052, $\pi/2$ )	-1.5236	(0, $\pi/2$ )	-26.719	(0.4052, 0)	1.5236
	-2	(0.3829, $\pi/2$ )	-1.5403	(0, $\pi/2$ )	-29.351	(0.3829, 0)	1.5403
7/8	2	(0.4889, $\pi/2$ )	-1.4994	(1, $\pi/2$ )	+46.042	(0.4889, 0)	1.4994
	1	(0.4778, $\pi/2$ )	-1.5004	(0, $\pi/2$ )	-46.035	(0.4778, 0)	1.5004
	0	(0.4666, $\pi/2$ )	-1.5048	(0, $\pi/2$ )	-48.235	(0.4666, 0)	1.5048
	-1	(0.4556, $\pi/2$ )	-1.5054	(0, $\pi/2$ )	-50.306	(0.4556, 0)	1.5054
	-2	(0.4446, $\pi/2$ )	-1.5086	(0, $\pi/2$ )	-52.571	(0.4446, 0)	1.5086

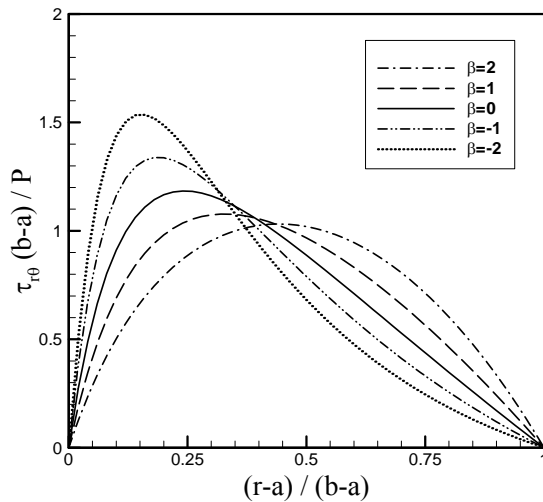
the beam. When the value of  $\beta$  is negative, the radial position of the maximum radial and shear stresses occurs near the inner surface of the beam. This is due to the fact that for a negative value of  $\beta$ , the stiffness of the beam decreases from inner surface to the outer one. On the contrary, when the inhomogeneity constant increases to a positive value, the stiffness of the outer surface increases and consequently the radial position of maximum stresses moves to outer surface of the beam. Another interesting point attracting ones attention is that the effect of inhomogeneity constant on the value and location of maximum stresses diminishes when the inner-to-outer radius ratio tends to unity (i.e. thin beam). In this case, the value of maximum non-dimensional shear stress approaches 3/2 and its location is at the middle surface, similar to the Euler-Bernulli beam. It is worth noting that the maximum values of tensile and compressive circumferential stresses always locate on the

surfaces of the beam. Thus, the variation of the inhomogeneity constant may cause the maximum value of tensile circumferential stress to be more than the value of maximum compressive circumferential stress and vice versa (Table 1).

In Fig. 2, the nondimensional normal circumferential stress is shown along the radial direction at  $\theta=\pi/4$ , for a thick FG curved beam with outer to inner radius ratio  $b/a=3$ . It is seen that for a homogeneous curved beam,  $\beta=0$ , there exists a maximum compressive value of circumferential stress at the inner radius of the beam which is more than the amount of the maximum tensile circumferential stress at the outer radius of the beam. By increasing the value of inhomogeneity constant in a FG curved beam, the intensity of compressive circumferential stress at the inner radius of the beam decreases. Regarding the curves of this figure, it can be seen that for a FG curved beam with  $\beta=2$ , the values of the maximum



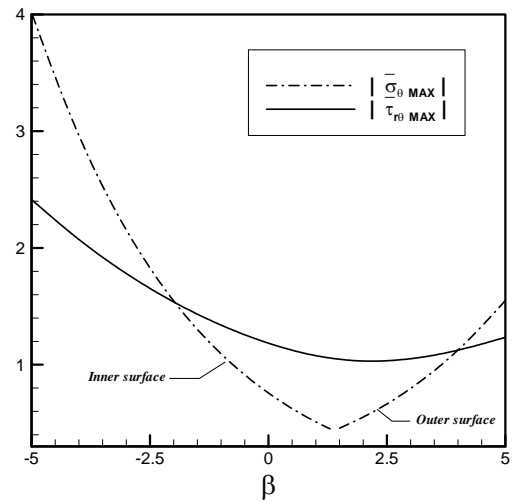
**Figure 2.** Variation of the non-dimensional circumferential stress along the radial direction for the FG curved beam



**Figure 3.** Variation of the non-dimensional shear stress along the radial direction for the FG curved beam

compressive circumferential stress at the inner edge and maximum tensile circumferential stress at the outer edge of the beam are approximately equal to each other.

The variation of non-dimensional in-plane shear stress across the radial direction is depicted in Fig. 3 for a thick FG curved beam with  $b/a=3$  at  $\theta=\pi/4$ . It can be seen that for a homogenous beam, the in-plane shear stress has asymmetric distribution in radial direction with respect to the middle surface



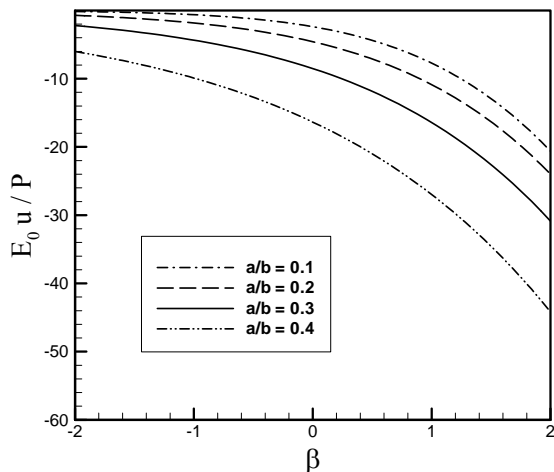
**Figure 4.** Variation of maximum values of the non-dimensional circumferential and shear stresses versus the inhomogeneity constant

of the beam. This asymmetric distribution of the in-plane shear stress is improved for the FG curved beam by increasing the amount of inhomogeneity constant. It is clear that for  $\beta=2$  the stress distribution along the radial direction is approximately symmetric. In other word, the peak of the stress occurs at the middle surface of the beam.

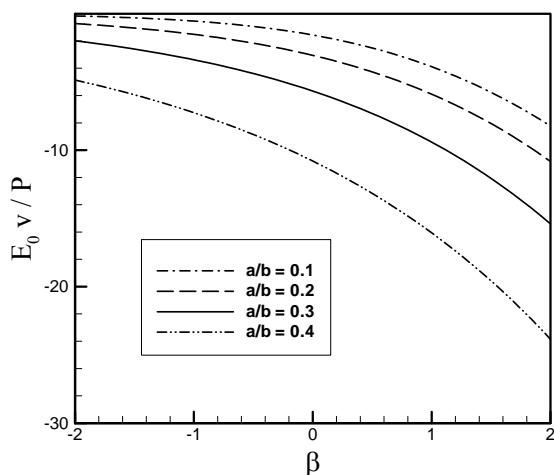
In Fig. 4, the variations of maximum values of the non-dimensional circumferential and shear stresses are depicted versus the inhomogeneity constant for a thick FG curved beam with  $b/a=3$ . It is observed that there exists an inhomogeneity constant,  $\beta$ , for which the maximum value of shear stress in the beam becomes minimum. Also, it is seen that by increasing the inhomogeneity constant, the maximum value of compressive circumferential stress in the beam (locates at the inner surface and  $\theta=\pi/2$ ) decreases, whereas the maximum value of tensile circumferential stress (locates at the outer surface and  $\theta=\pi/2$ ) increases. Thus, for a specific value of  $\beta$ , the maximum values of tensile and compressive circumferential stresses are equal.

Variations of non-dimensional radial and circumferential deflections of the upper end of the beam versus  $\beta$  are shown respectively in Figs. 5 and 6 for various inner-to-outer radius ratios. It can be seen that by increasing the inhomogeneity constant, both radial and circumferential





**Figure 5.** Variation of the radial deflection of the upper end of the beam versus the inhomogeneity constant



**Figure 6.** Variation of the circumferential deflection of the upper end of the beam versus the inhomogeneity constant

deflections of the beam increase. The reason for this is that the stiffness of the inner surface of the beam decreases when  $\beta$  increases. Also, it is concluded that as the inner-to-outer radius ratio increases, the deflection of the curved beam increases.

## 6. CONCLUSION

In this paper, a closed-form two-dimensional

elasticity solution has been developed for stresses and displacements of a functionally graded curved beam subjected to a shear force at its upper end. The main advantage of designing and analyzing such a beam, composed of functionally graded materials, is the possibility of improving the stress distribution profiles. It has been shown that for a specific value of the inhomogeneity constant, a proper stress distribution along the radial direction is obtained. Moreover, it has been found that all the stress and displacement relations converted to those of a homogenous curved beam, when the inhomogeneity constant approaches zero.

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