TECHNICAL NOTE

THE APPROACH OF CONVERGENCE TO STATIONARY STATE OF MULTI SERVER QUEUES WITH BALKING

Charan Jeet Singh*

Department of Mathematics, Guru Nanak Dev University
Amritsar, India
cjsmath@gmail.com

Madhu Jain

Department of Mathematics, Institute of Basic Science
Agra, India
madhujain@sancharnet.in

*Corresponding Author

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Abstract This investigation deals with multi server queues with balking. The customers arrive in poisson fashion and independent of time, under the assumption that the system is initially empty. The number of customers in the system stochastically increases and distributed in a steady state (stationary state) as required. The expressions for the measure of the speed of convergence from transient state to steady state within in the probability of the system are obtained.

Keywords Multi Server, Poisson, Stochastic, Steady State, Balking

چکیده این مقاله یک صف چند خدمت دهنده با ظرفیت محدود و بافر موقت را مورد بررسی قرار می‌دهد. مشتریان بر طبق توزیع پوآسن و مستقل از زمان – تحت شرایطی که سیستم در ابتدا خالی است – وارد می‌شوند. تعداد مشتریان در این سیستم بطور تصادفی افزایش می‌یابد و در حالت ثبات همانطور که مورد نیاز است، توزیع می گردند. عبارت ها برای اندازه گیری احتمال سرعت همگرایی از حالت گذرا به حالت ثبات تعریف می‌شوند.

1. INTRODUCTION

In performance evaluation of various queuing systems such as production, manufacturing, telecommunication, computer communication, etc., the stochastic process plays an important role. In multi server M/M/C queuing systems, the computation of time-dependent distribution attracts the interest of many researchers. Abate and Whitt [1] and Parthasarthy [11] discussed a simple approach for transient solution to Single-server M/M/1 queuing models. Baccelli and Massey [5] and Conolly Conolly and Langaris [6] also obtained the time dependent distribution for classical queuing system when the number of channel is unity. The transient analysis and other measures were obtained in multi server M/M/c (c > 1) queue by Kimura [9]. The numerical calculation of transient performance measures for the M/M/1 queue has been discussed by Abate and Whitt [2] and Van de Coevering [15]. Later, Stadje and Parthasarthy [14] obtained the expression for the measure of speed of convergence from transient state to stationary state in many server poisson queues. For detailed references, we refer to Kijima [8]. Due to wide applicability of discouragement queuing models, a large number of researchers have contributed in this direction and obtained various performance indices. Daley and Servi [7] proposed the method to obtain moments estimators for the arrival and customer's loss rates for many server queuing systems with a poisson arrival process and customer loss via balking. Artateljo and Lopez-Herrero [4] investigated ergodicity conditions making use of classical mean drift criterion for M/G/1 queues with balking. Mandelbaum and Shimkin [10] examined a model in which a customer arriving at an M/M/m queue which may not enter the system (i.e. balk), if he found that he will not get service immediately. The transient solution of non - truncated M/M/2 queue with balking and additional server for longer queues was suggested by Al-Seedy [3].

In this paper, transient analysis of a multi server M/M/C queuing model by incorporating balking was considered. This investigation facilitates the measure of speed of convergence towards a stationary state by including balking concepts in a multi server model which was not taken into consideration by earlier researchers. Without loss of generality, it is assumed that the service rate is equal to one. We derive the expression for the measure of convergence to a steady state of probability distribution. In Section 2, we have provided the mathematical formulation of the model and recursive relations for transient probabilities and steady state probabilities. The steady state moments are given in Section 3. The measure of speed of convergence from transient state to steady state, are based on moments, and by using integration techniques given in Section 4. In Section 5, we have defined the matrix method technique to verify the stationary probability under special cases. The numerical illustration for stationary probability is given in Section 6.

2. THE MATHEMATICAL MODEL AND RECURSIVE RELATIONS

We consider the M/M/C queuing model with balking. It is assumed that, the arrival rate of customers follows the poisson distribution with rateλ ($< c$) and without restriction of generality service rate to 1. Denote X(t): The number of customers in the system at time t.

$$\mu_i(t) = E[X(t)^i]\ \forall i = 1, 2, \ldots;$$

$$\mu_i = E[X^i]\ \forall i = 1, 2, \ldots$$

$$p_n(t) = P(X(t) = n):$$

Probability that there are n customers in the system at time t.

$$p_n = \lim_{t \to \infty} P_n(t):$$

Steady state probability of n customers being in the system.

The balking probability is given by

$$\beta_n = \begin{cases} 1, & n < c \\ \left(\dfrac{c}{n+1}\right), & n \geq c \end{cases}$$

The governing equations for the probabilities $P_n(t)$ are as follows:

$$\frac{d}{dt} p_0(t) = P_1(t) - \lambda\beta_0 P_0(t); n = 0 \tag{1}$$

$$\frac{d}{dt} p_n(t) = (n+1) P_{n+1}(t) - (\lambda\beta_n + n) P_n(t) + \lambda\beta_{n-1} P_{n-1}(t);\ 1 \leq n < c \tag{2}$$

$$\frac{d}{dt} p_n(t) = C P_{n+1}(t) - (\lambda\beta_n + c) P_n(t) + \lambda\beta_{n-1} P_{n-1}(t);\ n \geq c \tag{3}$$

The stationary probabilities P_n can be obtained as

$$p_n = \begin{cases} \dfrac{(c-1)!}{n!\lambda^{c-n-1}} P_{c-1}; & n = 0, 1, 2, \ldots, (c-2) \\ \dfrac{\lambda^{c-1}}{(c-1)!}\left[\dfrac{\lambda^c}{(c-1)!\,(c-\lambda)} + \sum_{j=0}^{(c-1)} \dfrac{\lambda^j}{j!}\right]^{-1}; & n = c-1 \\ \left(\dfrac{\lambda}{(n+1)}\right)^{n+1-c} P_{c-1}; & n = c, c+1, \ldots \end{cases} \tag{4}$$

Recursive relations for stationary state probabilities are

$$\overline{n+1}\,p_{n+1} - \bar{n}\,p_n = \lambda(\beta_n p_n - \beta_{n-1} p_{n-1}) \qquad (5)$$

$$\bar{n} p_n = \lambda\beta_{n-1} p_{n-1}$$

and

$$p_{-1} = 0 \qquad (6)$$

Where

$\bar{n} = \min(n, c)$.

3. STEADY STATE MOMENTS

In this section, we obtain the steady state moments in terms of stationary probabilities as follows:
Let

$$A_n = \frac{(c-1)!}{\lambda^{c-n-1}}$$

and

$$B_n = \frac{1}{n+1}\left(\frac{\lambda}{n+1}\right)^{n+1-c}$$

Since,

$$\sum_{n=0}^{c}(c-n)p_n = \sum_{n=0}^{\infty}(c-\bar{n})p_n = c-\lambda\left[\sum_{0}^{c-1}\frac{A_n}{n!}+1+c\sum_{c+1}^{\infty}B_n\right]p_{c-1} \qquad (7)$$

Now,

$$\sum_{n=0}^{c}(c-n)\,n\,p_n = \sum_{n=0}^{\infty}(c-\bar{n})\,n\,p_n = c\mu_1 - \sum_{n=1}^{\infty} n\,\lambda\,\beta_{n-1}p_{n-1} = c\mu_1 - \lambda\left[\sum_{1}^{c-1}\frac{A_n}{(n-1)!}+c+c\sum_{c+1}^{\infty} n\,B_n\right]p_{c-1} \qquad (8)$$

Again,

$$\sum_{n=0}^{c}(c-n)np_n = \lambda\sum_{n=0}^{c}(c-n)p_{n-1}\beta_{n-1} = \left[\lambda c\sum_{0}^{c-1}\frac{A_n}{n!} - \lambda\sum_{0}^{c-1}\frac{A_n}{(n-1)!}\right]p_{c-1} \qquad (9)$$

From Equations 8 and 9, we get

$$\mu_1 = \lambda\left[\sum_{0}^{c-1}\frac{A_n}{n!}+1+\sum_{c+1}^{\infty} n\,B_n\right]p_{c-1} \qquad (10)$$

Again,

$$\sum_{n=0}^{c}(c-n)n^2p_n = \sum_{n=0}^{\infty}(c-\bar{n})n^2p_n = c\mu_2 - \lambda\sum_{n=0}^{\infty}n^2\beta_{n-1}p_{n-1} = c\mu_2 - \lambda\left[\sum_{0}^{c-1}\frac{nA_n}{(n-1)!}+c^2+c\sum_{c+1}^{\infty}n^2B_n\right]p_{c-1} \qquad (11)$$

Also,

$$\sum_{n=0}^{c}(c-n)\,n^2p_n = \lambda\sum_{n=0}^{c}(c-n)\,np_{n-1}\beta_{n-1} = \lambda\left[c\sum_{0}^{c-1}\frac{A_n}{(n-1)!} - \sum_{0}^{c-1}\frac{nA_n}{(n-1)!}\right]p_{c-1} \qquad (12)$$

On comparing, Equations 11 and 12, we get

$$\mu_2 = \lambda\left[\sum_{0}^{c-1}\frac{A_n}{(n-1)!}+\sum_{c+1}^{\infty}n^2B_n\right]p_{c-1} \qquad (13)$$

4. THE METHOD OF INTEGRATION

In this section, we obtain the measure of convergence, based upon the moments as follows:

Let:

$$m_j(t) = p_j(t) - p_j$$

From Equations 1-3 and 5-6, we get

$$m'_j(t) = \overline{j+1}\, m_{j+1}(t) - \lambda\beta_j m_j(t) - \left[\bar{j} m_j(t) - \lambda\beta_{j-n} m_{j-n}(t)\right],\ j=0,1,2,...$$

Therefore,

$$\sum_{i=0}^{j} m'_j(t) = \overline{j+1}\, m_{j+1}(t) - \lambda\beta_j m_j(t) \quad (14)$$

First, we prove that

$$\mu'_1(t) = \sum_{j=0}^{c-1} (c-j) m_j(t) \quad (15)$$

Let,

$$h_j(t) = P(X(t) > j) = 1 - \sum_{i=0}^{j} p_i(t)$$

Since, the system is initially empty, we have $h_j = 0, \forall j$ and by Stochastic monotonicity, h_j is increasing function. So that, $h'_j(t) \geq 0$

Therefore,

$$\mu_1(t) = \sum_{j=0}^{\infty} h_j(t) = \sum_{j=0}^{\infty} \left(h_j(t) - h_j(0)\right) = \sum_{j=0}^{\infty} \int_0^t h'_j(s)ds = \int_0^t \sum_{j=0}^{\infty} h'_j(s)ds \quad (16)$$

($h'_j \geq 0$ and using monotone convergence theorem)
Now,

$$\sum_{j=0}^{\infty} h'_j(t) = \sum_{j=0}^{\infty} \frac{d}{dt}\left(1 - \sum_{i=0}^{j} p_i(t)\right) = -\sum_{j=0}^{\infty} \sum_{i=0}^{j} m'_i(t)$$

Using Equation 14, we get

$$\sum_{j=0}^{\infty} h'_j(t) = \lambda \sum_{j=0}^{\infty} \beta_j m_j(t) - \sum_{j=0}^{\infty} \overline{j+1}\, m_{j+1}(t)$$

Since $\sum_{j=0}^{\infty} m_j(t) = 0$, therefore

$$\sum_{j=0}^{\infty} h'_j(t) = -\sum_{j=0}^{\infty} \bar{j} m_j(t) = \sum_{j=0}^{c-1} (c-j) m_j(t) \quad (17)$$

On putting the value of $\sum_{j=0}^{\infty} h'_j(t)$ from Equation 17 in Equation 16 and using the fundamental theorem of calculus, under the assumption that $\sum_{j=0}^{\infty} h'_j(t)$ is a continuous function of t, we get Equation 15.

For special cases when $\beta_n = \beta$, the measure of convergence is obtained, based upon the moments. On the lines of Stadje and Parthasarathy [14], we obtain

$$D = \int_0^{\infty} \left(\mu_1 - \mu_1(t)\right) dt = \frac{\mu_1 + \mu_2}{2(c - \lambda\beta)} + \frac{1}{(c - \lambda\beta)} \sum_{j=0}^{c-1} j(c-j) \int_0^{\infty} \left(p_j - p_j(t)\right) dt \quad (18)$$

When c = 1, i.e. for single server model, Equation 18 reduces to

$$D = \frac{\mu_1 + \mu_2}{2(1 - \lambda\beta)} \quad (18\ a)$$

When c = 2, i.e. for two server model, Equation 18 yields

$$D = \frac{\mu_1 + \mu_2}{2(2 - \lambda\beta)} - \frac{\pi_1}{2 - \lambda\beta} \left[\frac{1}{\lambda\beta} - \pi_1 \left(\frac{1}{\lambda^2\beta^2} + \frac{4\lambda\beta(2 + \lambda\beta)}{(2 - \lambda\beta)^3}\right)\right] \quad (18\ b)$$

Where,

$$\pi_1 = \frac{\lambda\beta(2-\lambda\beta)}{(2+\lambda\beta)}.$$

5. THE MATRIX METHOD

For the particular case, when $\beta_n = \beta$, the (c-1) x (c-1) matrix is considered, which has the form:

A =

$$\begin{bmatrix} -\lambda\beta & 1 & 0 & 0 & \dots & 0 \\ \lambda\beta & -(\lambda\beta+1) & 2 & 0 & \dots & 0 \\ 0 & \lambda\beta & -(\lambda\beta+2) & 3 & \dots & 0 \\ : & \dots & \dots & \dots & \dots & \dots \\ : & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \lambda\beta & -(\lambda\beta+c-2) \end{bmatrix}$$

Since, we have

$$(sI-A)^{-1} = \left(a_{kj}(s)\right); \quad k, j = 0,1,\dots,c-2.$$

following Parthasarathy and Sharafali [12], the following equation is obtained:

$$a_{kj}(s) = \begin{cases} \dfrac{u_{c-1,j+1}(s)u_{k,0}(s) - u_{k,j+1}(s)u_{c-1,0}(s)}{(j+1)u_{c-1,0}(s)}; j = 0,1,\dots,c-3 \\ \dfrac{u_{k,0}(s)}{u_{c-1,0}(s)}; j = c-2 \end{cases} \tag{19}$$

Where the functions $u_{k,j}(s)$ are recursively determined by

$$u_{k,j}(s) = 0; j > k.$$

$$u_{k,k}(s) = 1; k = 0,1,\dots,c-2.$$

$$u_{k+1,k}(s) = \frac{s+\lambda\beta+k}{k+1}; \quad k = 0,1,\dots,c-3.$$

$$u_{k-1,k-j}(s) = \frac{(s+\lambda\beta+k)u_{k,k-j}(s) - \lambda\beta u_{k-1,k-j}(s)}{k+1}; \quad j \le k, k = 1,\dots,c-3.$$

$$u_{c-1,j}(s) = \begin{cases} (s+\lambda\beta+c-2)u_{c-2,j}(s) - \lambda\beta u_{c-3,j}(s); j = 0,1,\dots,c-3. \\ s+\lambda\beta+c-2; j = c-2. \end{cases}$$

Let

$$B_{k,n} = \frac{1}{k} + \frac{\lambda\beta}{k(k-1)} + \dots + \frac{(\lambda\beta)^n}{k(k-1)\dots(k-n)}; \quad n = 0,1,\dots,k-1$$

The recursion can be written as:

$$u_{k,j}(s) = \frac{(\lambda\beta)^{k-j}}{k(k-1)\dots(j+1)} + jB_{k,k-j-1} + s\sum_{n=j}^{k-1} B_{k,k-j-1}u_{n,j}(s); \quad k = 1,2,\dots,(c-2) \text{ and } j = 0,1,\dots,(k-1) \tag{20}$$

$$u_{c-1,j}(s) = \frac{(\lambda\beta)^{c-j-1}}{(c-2)(c-3)\dots(j+1)} + j(c-1)B_{c-1,c-j-2} + s(c-1)\sum_{n=j}^{c-2} B_{c-1,c-2-n}u_{n,j}(s); \quad j = 0,1,\dots,(c-2) \tag{21}$$

Suppose $f_j(s) = \sum_{k=0}^{c-2} a_{kj}(s)$ and $f_0(s)$ and $f_{c-2}(s)$ are needed to be found , the sums of first and of the last columns of.$[(sI - A)]^{-1}$.

On putting s = 0,

$$f_0(0) = \left[u_{c-1,1}(0) \sum_{k=0}^{c-2} u_{k,0}(0) - u_{c-1,0}(0) \sum_{k=1}^{c-2} u_{k,1}(0) \right] / u_{c-1,0}(0) \qquad (22)$$

Where

$$u_{k,0}(0) = \frac{(\lambda\beta)^k}{k!}; k = 0,1,...,(c-2).;$$

$$u_{c-1,0}(0) = \frac{(\lambda\beta)^{c-1}}{(c-2)!}$$

$$u_{k,1}(0) = \frac{(\lambda\beta)^{k-1}}{k!} + B_{k,k-2}; k = 2,3,...,(c-2).;$$

$$u_{c-1,1}(0) = \frac{(\lambda\beta)^{c-2}}{(c-2)!}$$

and

$$f_{c-2}(0) = \left(1 + \lambda\beta + ... + \frac{(\lambda\beta)^{c-2}}{(c-2)!} \right) / \frac{(\lambda\beta)^{c-1}}{(c-2)!} \qquad (23)$$

Also

$$f'_{c-2}(0) = \frac{\left[u_{c-1,0}(0) \sum_{k=0}^{c-2} u'_{k,0}(0) - u'_{c-1,0}(0) \sum_{k=0}^{c-2} u_{k,0}(0) \right]}{\left(u_{c-1,0}(0) \right)^2}; \quad k = 0,1,...,(c-2) \qquad (24)$$

Where

$$u'_{k,0}(0) = \sum_{n=0}^{k-1} B_{k,k-n-1} \frac{(\lambda\beta)^n}{n!},$$

$$u'_{c-1,0}(0) = (c-1) \sum_{n=0}^{c-2} B_{c-1,c-n-2} \frac{(\lambda\beta)^n}{n!}$$

Note that the stationary probability is obtain as

$$p_{c-1} = \left[\frac{\lambda\beta}{c - \lambda\beta} + \frac{(c-1)!}{(\lambda\beta)^{c-1}} \sum_{i=0}^{c-1} \frac{(\lambda\beta)^i}{i!} \right]^{-1} \qquad (25)$$

This value of stationary probability P_{c-1} coincides with Equation 4 under special case when $\beta_n = \beta$.

6. NUMERICAL ILLUSTRATION

To check the validity of the investigation, we take the numerical illustration and provide the necessary computation for stationary probabilities. The computational results are summarized in the form of a table. Table 1 displays the stationary probability P_{c-1} for c = 1, 2 for different arrival rates of customers λ = 1, 2, 3, 4, 5 and balking probabilities β = .05, .10, .15. It is observed that for single server models c = 1, P_0 decreases as λ and β increase. However, for multi server models when we fix c = 2, P_{c-1} increases as λ and β increase, except for high traffic loads when λ = 5 where it first decreases and then increases as we increase β.

7. DISCUSSION

In this investigation, we have obtained the steady state moments and measure of convergence for multi server queue by incorporating balking. Our study finds applications in performance evaluation of queuing systems in particular computer and telecommunication systems where transient states as well as steady states play key roles in stochastic modeling in order to maintain the desired grade of

TABLE 1. Evaluation of Stationary Probability P_{C-1}.

β \ λ	1	2	3	4	5
c = 1					
0.05	0.95	0.90	0.85	0.80	0.75
0.10	0.90	0.80	0.70	0.60	0.50
0.15	0.85	0.70	0.55	0.40	0.25
c = 2					
0.05	0.04	0.09	0.12	0.16	0.54
0.10	0.09	0.16	0.22	0.27	0.30
0.15	0.12	0.22	0.28	0.32	0.34

service.

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8. REFERENCES

1. Abate, J. and Whitt, W., "Transient behavior of the M/M/1 queue starting at the origin", ***Queuing Systems***, 2, (1987), 41-65.
2. Abate, J. and Whitt, W., "Calculating time dependent performance measures for the M/M/1 queue, ***IEEE Trans. Comm***., 37, (1989), 1102-1104.
3. Al-Seedy, R. O., "A transient solution of non-truncated queue M/M/2 with balking and additional server for longer queue", ***Appl. Math. Comp***., Vol. 15, No. 3, (2004), 763-769.
4. Artateljo, J. R. and Lopez-Herrero, M. J., "On the single server retrial queue with balking", Inform. Sci., Vol. 38, No. 1, (2000), 34-50.
5. Baccelli, F. and Massey, W. A., "A sample path analysis of the M/M/1 queue", ***Appl. Prob***., 26, (1989), 418-422.
6. Conolly, B. W. and Langaris, C., "On a new formula for the transient state probabilities for M/M/1 queue and computational implications", ***Appl. Prob***., 30, (1993), 237-246.
7. Daley, D. J. and Servi, L. D., "Idle and busy periods in stable M/M/K queues", ***Appl. Prob***., 35, (1998), 950-962.
8. Kijima, M., "A Markov Processes for Stochastic Modeling", Chapman and Hall, London, (1997).
9. Kimura, T., "Approximation for multi server queues system interpolations", ***Queuing Systems***, 17, (1994), 347-382.
10. Mandelbaum, A. and Shimkin, N., "A model for rational abandonment from invisible queues", ***Queuing System***, 36, (2001), 141-173.
11. Parthasarathy, P. R., "A transient solution to an M/M/1 queue: a simple approach", ***Adv. Appl. Prob.***, 19, (1987), 997-998.
12. Parthasarathy, P. R. and Sharafali, M., "Transient solution of the many server poisson queue-a simple approach", ***Appl. Prob***., 26, (1989), 584-596.
13. Raju, S. N. and Bhat, U. N., "A computationally oriented analysis of the GI/M/1 queue", Opsearch, 19, (1982), 67-83.

14. Stadje,W. G. and Parthasarathy, P. R., "On the convergence to stationarity of the many server poisson queue", ***Appl. Prob***., 36, (1999), 546-557.

15. Van De Coevering, M. C. T., "Computing transient performance measures for the M/M/1 queue", OR Spekturm, 17, (1995), 19-22.