TECHNICAL NOTE

G/G/r MACHINE REPAIR PROBLEM WITH SPARES AND ADDITONAL REPAIRMAN

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Abstract The machine repair problem with spares and additional repairman is analyzed. The inter-failure and repair times of the units are general identical and independently distributed. The failure and repair rates are assumed to be state dependent. Using diffusion approximation technique, we obtain the queue size distribution under steady state. The average number of failed units, average number of operating units in the system and the probability of the system being short are obtained by using the queue size distribution.

Key Words Queue, Diffusion, Machine Repair, Warm Standby, Additional Repairman, State Dependent Rate

چکیده مساله تعمیر ماشین با لوازم یدکی و تعمیر کارهای اضافی در این تحقیق مورد تجزیه و تحلیل قرار می‌گیرد. زمانهای بین خرابی و زمان تعمیر هر واحد دارای توزیع همشکل و مستقل از یکدیگرند. نرخهای خرابی و تعمیر بنا به فرض مستقل از وضعیت کاری هستند. با استفاده از تکنیکهای تقریب پخش، به یک سیستم صف در حالت پایدار می‌رسیم. میانگین تعداد وسایل خراب در سیستم و همچنین احتمال اینکه سیستم کمبود داشته باشد، با استفاده از توزیع سیستمهای صف در این تحقیق محاسبه می‌گردد.

INTRODUCTION

An interruption in machining system during the operation not only affects the quality of manufactured product but also increases the cost of production. The problem of machine repair and automation is to employ the right number of repairmen to look after a certain number of machines. In many industrial processes where machines work, the problem of providing spare machines may arise frequently. The provision of spares and additional repairman may improve the running efficiency and operating utilization of the machining system having multi-components. Gross et al. [1] studied a queuing model for spare provisioning. Sivazlian and Wang [2] gave the economic analysis of the M/M/R machine repair problem with warm standby spares. Jain [3] considered the M/M/R machine repair problem with spares and additional repairmen by using queue size distribution of failed machines. Jain et al. [4] obtained the steady state queue size distribution for M/M/C/K/N machine repair problem with balking, reneging, spares and additional repairman. Ching [5] gave the Markovian approximation for manufacturing systems of unreliable machines in tandem.

Exact solution to the queuing problems with inter-arrival time and service time drawn from arbitrary distributions are difficult. In such cases we are interested to get the approximate solution by using the diffusion approximation technique. By introducing diffusion parameters and accurate boundary conditions to the underlying diffusion process, the complex machine repair problems can be analyzed. Various papers have been devoted for multi-server queuing problem by using diffusion process but most of them dealt with infinite calling population. Sivazlian and Wang [6] considered the G/G/R machine repair problem with warm standby system using the diffusion technique. Jain [7] gave some performance measures for a $G^x/G/m$ machine interference problem with spare machines by using diffusion process with reflecting boundaries. Jain [8] studied (m, M) machine repair problem with spares and state dependent rates. The problem was solved by diffusion approximation technique with reflecting boundaries. Jain and Singh [9] developed diffusion process for optimal flow control of a G/G/c finite capacity queue. They obtained the mean throughput under steady state.

This paper is concerned with diffusion approximation technique for the problem of machine repair system with spare units and one additional repairman. If a unit fails, a spare unit replaces it. The failed unit is sent immediately for repairing to service facility having permanent and one specialized additional repairman. The repairman can repair only one failed unit at a time. When all permanent repairmen are busy, the repair rate is faster in comparison to the rate when at least one repairman is idle. The steady state queue size distribution for the number of failed units is established by using the means and variances of life time and repair time distributions.

MODEL DESCRIPTION

We consider G/G/r machine repair problem with spares as follows: There are M operating units, S spare units as standby, r permanent repairmen and one additional repairman in the repair facility. Whenever a unit fails, it is immediately sent to repair facility where it is repaired in the order of breakdowns i. e. FCFS. Each repairman can repair only one failed unit at a time. If all repairmen are busy, the failed units must wait until a repairman is available. The failed unit is replaced by spare provided any spare unit is available in the standby group. If at least m out of M units is in operating group then the system will be in operating mode. When the repairing of a failed unit is completed, it is as good as new one. The repaired unit goes into the operating group if there are less than M units otherwise it goes into the standby group. If all spare units are being used and a breakdown occurs, the system is said to be short, and work with degraded rate. In the short system if more than M-m units fail in operating group, the system breaks down. The inter-failure time and repair time of the units are general identical and independently distributed. The failure rates of operating units and spare units are different. As soon as the number of failed units reaches N, the additional repairman is turned on.

The state dependent failure and repair rates for the model are as follows:

$$\lambda_n = \begin{cases} M\lambda + (S-n)\alpha & 0 \le n \le S \\ (M+S-n)\lambda_1 & S < n \le M+S-m \end{cases} \tag{1}$$

$$\mu_n = \begin{cases} n\mu_0 & 1 \le n < r \\ r\mu & r \le n < N \\ r\mu + \mu_1 & N \le n \le M+S-m \end{cases} \tag{2}$$

where λ and λ_1 are the mean failure rates of operating units in normal and short system respectively; α is the mean failure rate of spare units, μ_0 is the mean repair rate when at least one permanent repairman is idle, μ is the mean repair rate when all permanent repairmen are busy and μ_1 mean repair rate for additional repairman.

DIFFUSION PROCESS

To model machine repair problem, we use diffusion process with reflecting boundaries at x=0, x=M+S-m. Let $p_n(t)$ denote the probability that there are n units in the system at time t. The continuous probability density function $p_n(x,t)$ is taken corresponding to discrete probability mass function $p_n(t)$. The p(x,t) satisfies the following Fokker - Planck equation (see Cox and Miller, [10])

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\{p(x,t)b(x,t)\} - \frac{\partial}{\partial x}\{p(x,t)a(x,t)\} \tag{3}$$

where a(x, t) and b(x, t) are respectively the drift and variance of diffusion process at time t.

Using reflecting boundaries at x = 0 and x=M+S-m, the solution of Equation 3 under steady state is given by

$$p(x) = \frac{\text{Constant}}{b(x)} \exp\left\{ 2\int_0^x \frac{a(x)}{b(x)} dx \right\} \qquad (4)$$

Here a(x) and b(x) are the steady state drift and variance respectively. The square coefficients of variation of inter-failure and repair time distributions are denoted by C_a^2 and C_s^2 respectively. Now we propose the steady state drift a(x) and variance b(x)for our model as follows:

Case I: $r \le S$

1. For $0 \le x < r$

$a(x) = [M\lambda + (S-x)\alpha] - x\mu_0$

$b(x) = [M\lambda + (S-x)\alpha] C_a^2 + x\mu_0 C_s^2$

2. For $r \le x \le S$

$a(x) = [M\lambda + (S-x)\alpha] - r\mu$

$b(x) = [M\lambda + (S-x)\alpha] C_a^2 + r\mu C_s^2$

3. For $S < x < N$

$a(x) = (M+S-x)\lambda_1 - r\mu$

$b(x) = (M+S-x)\lambda_1 C_a^2 + r\mu C_s^2$

4. For $N \le x \le M + S - m$

$a(x) = (M+S-x)\lambda_1 - (r\mu + \mu_1)$

$b(x) = (M+S-x)\lambda_1 C_a^2 + (r\mu + \mu_1) C_s^2$

Case 2: $S < r$

1. For $0 \le x \le S$

$a(x) = M\lambda + (S-x)\alpha - x\mu_0$

$b(x) = [M\lambda + (S-x)\alpha] C_a^2 + x\mu_0 C_s^2$

2. For $S < x < r$

$a(x) = (M+S-x)\lambda_1 - x\mu_0$

$b(x) = (M+S-x)\lambda_1 C_a^2 + x\mu_0 C_s^2$

3. For $r \le x < N$

$a(x) = (M+S-x)\lambda_1 - r\mu$

$b(x) = (M+S-x)\lambda_1 C_a^2 + r\mu C_s^2$

4. For $N \le x \le M + S - m$

$a(x) = (M+S-x)\lambda_1 - (r\mu + \mu_1)$

$b(x) = (M+S-x)\lambda_1 C_a^2 + (r\mu + \mu_1) C_s^2$

THE QUEUE SIZE DISTRIBUTION

We obtain the steady state queue size distribution p(x) by using Equation 4 and the values of a (x) and b(x) for different intervals as follows:

Case 1: $r \le S < N$

$$p(x) = \begin{cases} C g_1(x) \equiv p_1(x), & o \le x < r \\ C \dfrac{g_1(r)}{g_2(r)} g_2(x) \equiv p_2(x), & r \le x \le S \\ C \dfrac{g_1(r) g_2(S)}{g_2(r) g_3(S)} g_3(x) \equiv p_3(x), & S < x < N \\ C \dfrac{g_1(r) g_2(S) g_3(N)}{g_2(r) g_3(S) g_4(N)} g_4(x) \equiv p_4(x), & N \le x \le M + S - m \end{cases} \qquad (5)$$

where C is a constant and

$$g_1(x) = \left(x - \frac{\beta}{\alpha - \mu_0\eta} \right)^{\left[2\beta\mu 0(1+\eta)/(\mu 0\eta - \alpha)^2 C_a^2\right] - 1} \times$$

$$\exp\left\{ \frac{-2(\alpha + \mu_{0)} x}{(\mu_0\eta - \alpha) C_a^2} \right\} \qquad (6)$$

$$g_2(x)=\left(x-\frac{\beta+r\mu\eta}{\alpha}\right)^{\left[2r\mu(1+\eta)/\alpha C_a^2\right]-1}\exp\left\{\frac{2x}{C_a^2}\right\}$$

(7)

$$g_3(x)=\left(x-\frac{\gamma+r\mu\eta}{\lambda_1}\right)^{\left[2r\mu(1+\eta)/\lambda_1 C_a^2\right]-1}\exp\left\{\frac{2x}{C_a^2}\right\}$$

(8)

$$g_4(x)=\left(x-\frac{\gamma+(r\mu+\mu_1)\eta}{\lambda_1}\right)^{\left[2(r\mu+\mu_1)(1+\eta)/\lambda_1 C_a^2\right]-1}\exp\left\{\frac{2x}{C_a^2}\right\}$$

(9)

where

$$\beta=M\lambda+S\alpha,\ \eta=\frac{C_s^2}{C_a^2},\ \gamma=(M+S)\lambda_1 \qquad (10)$$

The constant C can be determined by using the normalizing condition

$$\int_0^r p_1(x)dx+\int_r^S p_2(x)dx+\int_S^N p_3(x)dx+\int_N^{M+S-m} p_4(x)dx=1$$

(11)

The value of p_n is approximated by

$$\hat{P}_n=\int_{n-1/2}^{n+1/2}p(x)dx \qquad 1\leq n\leq M+S-m \qquad (12)$$

To obtain the value of p_{M+S-m}, we propose the following approximation

$$\hat{P}_{M+S-m}=\int_{M+S-m-1/2}^{M+S-m}p_4(x)dx \qquad (13)$$

Case 2: S<r<N

In this case the steady state queue size distribution p(x) is determined by using Equation 4 and the values of a(x) and b(x) for different intervals as follows:

$$p(x)=\begin{cases} C'g_1(x)\equiv p_1(x), 0\leq x\leq S \\ C'\dfrac{g_1(S)}{g_5(S)}g_5(x)\equiv p_5(x), S<x<r \\ C'\dfrac{g_1(S)g_5(r)}{g_5(S)g_3(r)}g_3(x)\equiv p_6(x), r\leq x<N \\ C'\dfrac{g_1(S)g_5(r)g_3(N)}{g_5(S)g_3(r)g_4(N)}g_4(x)\equiv p_7(x), \\ \qquad N\leq x\leq M+S-m \end{cases}$$

(14)

where C' is an arbitrary constant which can be determined by using the normalizing condition and

$$g_5(x)=\left(x-\frac{\gamma}{\lambda_1-\mu_0\eta}\right)^{\left[\frac{2\gamma\mu_0(1+\eta)}{(\mu_0\eta-\lambda_1)^2C_a^2}\right]-1}\times$$

$$\exp\left\{\frac{-2(\lambda_1+\mu_0)x}{(\mu_0\eta-\lambda_1)C_a^2}\right\} \qquad (15)$$

SOME PERFORMANCE INDICES

Denoting the average number of failed units in the system by L_k (k = 1 for case 1 and k =2 for case 2), we have

$$\begin{aligned}L_1=&\sum_{n=1}^{M+S-m}n\hat{p}_n=\sum_{n=1}^{r-1}n\int_{n-1/2}^{n+1/2}p_1(x)dx+r\left[\int_{r-1/2}^{r}p_1(x)dx+\right.\\&\left.\int_r^{r+1/2}p_2(x)dx\right]+\sum_{n=r+1}^{S-1}n\int_{n-1/2}^{n+1/2}p_2(x)dx+S\left[\int_{S-1/2}^{S}p_2(x)dx+\right.\\&\left.\int_S^{S+1/2}p_3(x)dx\right]+\sum_{n=S+1}^{N-1}n\int_{n-1/2}^{n+1/2}p_3(x)dx+N\left[\int_{N-1/2}^{N}p_3(x)dx+\right.\\&\left.\int_N^{N+1/2}p_4(x)dx\right]+\sum_{n=N+1}^{M+S-m-1}n\int_{n-1/2}^{n+1/2}p_4(x)dx+\\&(M+S-m)\int_{M+S-m-1/2}^{M+S-m}p_4(x)dx\end{aligned}$$

(16)

$$L_2 = \sum_{n=1}^{M+S-m} n\hat{p}_n = \sum_{n=1}^{S-1} n\int_{n-1/2}^{n+1/2} p_1(x)dx + S\left[\int_{S-1/2}^{S} p_1(x)dx + \int_{S}^{S+1/2} p_5(x)dx\right] + \sum_{n=S+1}^{r-1} n\int_{n-1/2}^{n+1/2} p_5(x)dx + r\left[\int_{r-1/2}^{r} p_5(x)dx + \int_{r}^{r+1/2} p_6(x)dx\right] + \sum_{n=r+1}^{N-1} n\int_{n-1/2}^{n+1/2} p_6(x)dx + N\left[\int_{N-1/2}^{N} p_6(x)dx + \int_{N}^{N+1/2} p_7(x)dx\right] + \sum_{n=N+1}^{M+S-m-1} n\int_{n-1/2}^{n+1/2} p_7(x)dx + (M+S-m)\int_{M+S-m-1/2}^{M+S-m} p_7(x)\,dx \tag{17}$$

The average number of operating units in the system is denoted by $E(O_k)$ and is obtained as

$$E(O_k) = M\sum_{n=0}^{S}\hat{p}_n + \sum_{n=S+1}^{M+S-m}(M+S-n)\hat{p}_n, k = 1,2 \tag{18}$$

The probability of system being short is given by

$$\hat{p}_S = \sum_{n=S+1}^{M+S-m}\hat{p}_n \tag{19}$$

The probability of system failure is

$$\hat{p}_F = \hat{p}_{M+S-m} = \int_{M+S-m-1/2}^{M+S-m} p_i(x)dx \tag{20}$$

where i = 4, for case 1 and i = 7 for case 2.

The average number of spare units in the system is

$$E(S) = \sum_{n=0}^{S}(S-n)\hat{p}_n \tag{21}$$

The average number of idle repairmen is given by

$$E(I) = \sum_{n=0}^{r-1}(r-n)\hat{p}_n \tag{22}$$

CONCLUSION

We have modeled the machine repair problem with spares and additional repairman as diffusion process. The steady state queue size distribution is established in terms of drift and variance of the diffusion process. The incorporation of state dependent rates makes our model more realistic in comparison to the existing model. By providing additional repairman, the system availability can be improved to a great extent. The results provided can be easily computed by taking numerical integration. The present model can be extended to bulk failure/repair problem, which is subject of our further research.

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