A NEW APPROACH TO APPROXIMATE COMPLETION TIME DISTRIBUTION FUNCTION OF STOCHASTIC PERT NETWORKS

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(Received: December 9, 1998 – Accepted in Final Form: October 5, 2001)

Abstract The classical PERT approach uses the path with the largest expected duration as the critical path to estimate the probability of completing a network by a given deadline. However, in general, such a path is not the most critical path (MCP) and does not have the smallest estimate for the probability of completion time. The main idea of this paper is derived from the domination structure between paths that was presented by Soroush for the first time. This paper develops this domination structure and its properties, which make Soroush's algorithm work faster in some cases. Then a labeling algorithm is presented that is able to compute the MCP from starting node of the network to any node of the network. Also, suitable and practical completion time distribution function estimation is defined. In many cases, the estimation is obtained by the developed method is better than that of Soroush's. To clarify the point, some examples are given. Finally, conclusions are presented.

Key Words PERT, Most Critical Path, Stochastic Network, Completion Time, Distribution Function, Approximation

چکیده رویکرد پرت کلاسیک از مسیری که نسبت به سایر مسیرها بزرگترین ارزش انتظاری مدت زمان را دارد بهره می گیرد تا احتمال تکمیل شبکه را در ظرف موعد مشخص تخمین بزند. به هر حال، در حالت کلی، چنین مسیری محتمل ترین مسیر بحرانی (MCP) نیست و از کوچکترین تخمین برای احتمال زمان تکمیل برخوردار نیست. نقطه نظر عمده این مقاله از ساختار غالب مابین مسیرها که برای نخستین بار سروش آن را مطرح کرده است نشئت می گیرد. این مقاله این ساختار غالب و خواص آن را توسعه می دهد که موجب می شود در مواردی الگوریتم سروش سریعتر کار کند. سپس یک الگوریتم برچسب زنی ارائه می شود که محاسبه MCP را از گره شروع شبکه تا هر گره ای در شبکه امکان پذیر می سازد. همچنین، تخمینی از تابع توزیع زمان تکمیل عملی و مناسب تعریف می شود. در مواردی، تخمین حاصله از روش توسعه یافته بهتر از تخمین سروش است. برای اینکه موضوع واضح شود، مثالهایی عنوان شده است. نهایتا، نتایج ارائه می شوند.

INTRODUCTION

A PERT network is an acyclic, connected and directed graph. The network has one starting and one terminal node. PERT networks are useful models for project planning and control. Duration of all activities is positive random variables with known probability distribution. The completion time of the project is a random variable whose realization can be determined, however its exact distribution function (F(t)) is very difficult to calculate for most PERT networks. The difficulty in evaluating F(t) that stems from the interdependency and large number of the paths in the networks, has motivated many studies. Harthley and Wortham [1] consider block seriesparallel networks of mini-networks. They allow the mini-networks to be a single arcs or Wheatsone bridge type of networks. They develop formulae to replace a Wheatsone bridge type of network by a single arc with appropriate distribution function. Ringer [2] generalizes Harthley and Wortham's work by allowing double Wheatsone bridges as mini-networks. When all activity durations are exponential, PERT networks are formulated as a continuous time Markov chain (CTMC), and the project completion time can be thought of as a particular first passage time in this CTMC. Kulkarni and Adlakha [3] have taken this approach. Martin [4] has provided a systematic way of implementing convolution and multiplication operations, where probability distribution of each activity is polynomial.

Dodin [5] derives a bound for F(t) with the assumption of independent random variables at network. Robillard and Trahan [6] derive a lower bound using Laplace transforms, assuming that activity times are independent. Also, Kleindorfer [7] provides upper and lower bounds for F(t) where discrete random variables are considered. Van Slyke [8] estimates the mean of F(t) by means of critical index of paths. Sculli [9] derives an approximation for the mean and variance of F(t) where normal probability distributions are considered.

Kamburowski [10] obtains upper and lower bounds for mean and lower bounds for variance of F(t). Fulkerson [11], Clingen [12], Elmaghraby [13], Robillard and Trahan [14], Lindsey [15], Dodin [16], derive bounds for the mean of F(t) for some cases.

About MCP identification, Martin [4] defines criticality index of a path but does not present a method for its computation. Sigal, Pritsker and Solberg [17] derive a method that stems from Monte Carlo simulation. Also, Fisher and Goldstein [18] present an algorithm to calculate it. Elmaghraby and Dodin [19] discuss criticality of activities. Dodin [20] derives a method to identify K most critical path in a network. In the next sections, because of dependency of our approach on Soroush's algorithm [21], his work is discussed and the domination structure presented by him is developed. Then a labeling algorithm for MCP identification is derived at a given time. At the end, a new approach on approximate F(t) based on MCP is derived. Finally, numerical analysis is made and conclusion is presented.

PROBLEM IDENTIFICATION

Let G (V,A) be a PERT network where V is the set of nodes and A is the set of arcs and $|V| = n, |A| = m$. Let $R = \{r_1, \ldots, r_{|R|}\}$ is the set of paths between v_1 and v_n. The network is stochastic in the sense that all activity times t_k, $a_k \in A$ are random variables. The duration T^* of the PERT network is;

$$T^* = \max_{r_j \in R}\{T_j\}, \text{ where } T_j = \sum_{a_k \in r_j} t_k, r_j \in R \quad (1)$$

Then, the probability to complete the project by a given deadline t, is;

$$p\{T^* \leq t\} = p\left\{\max_{r_j \in R}\{T_j\} \leq t\right\} = p\{T_j \leq t, \forall r_j \in R\} \quad (2)$$

To determine an upper bound on $P\{T^* \leq t\}$, let us rewrite (2) as:

$$p\{T^* \leq t\} = p\{T_i \leq t\} p\{T_j \leq t, \forall r_j \neq r_i \in R \mid T_i \leq t\}, r_i \in R \quad (3)$$

Utilizing probability theory and central limit theorem we obtain:

$$p\{T^* \leq t\} \leq p\{T_i \leq t\} = p\{Z \leq z_i\}, \quad r_i \in R \quad (4)$$

where Z is a standard normal variable, and

$$z_i = z(\mu_i, \sigma_i) = \frac{t - \mu_i}{\sigma_i}, \mu_i \geq 0, \sigma_i > 0, r_i \in R; \mu_i = \sum_{a_k \in R_i} \mu_k, \sigma_i = \left[\sum_{a_k \in R_i} \sigma_k^2\right]^{1/2} \quad (5)$$

In order to obtain the best upper bound from (4), we define MCP as an $r^* \in R$ such that,

$$p\{Z \leq z_{r^*}\} = \min_{r_i \in R}\{Z \leq z_i\} \text{ or if } z_{r^*} = \min_{r_i \in R}\{z_i\}$$

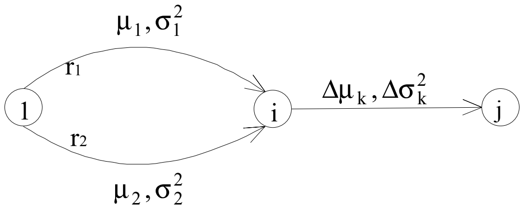

Figure 1. A PERT network with means and standard deviations for paths.

This path provides the least upper bound estimate for $P\{T^* \le t\}$. Meanwhile, assuming r_1, $r_2 \in R$, and a project deadline t, Soroush defines that r_1 stochastically dominates r_2, written as:

$r_1 \succ r_2$, if $p\{Z \le z_1\} < p\{Z \le z_2\}$, or if $z_1 < z_2$.

MCP IDENTIFICATION

In order to avoid complex integrations to compute (4), Soroush [21] establishes a domination structure between paths by which MCP is identified. Now, consider the PERT network of Figure 1.

He shows that if $r_1 \succ r_2$ then $z_1 < z_2$, but this relation will no longer hold when r_1 and r_2 are augmented by Δr_k. Let $\sigma_1 < \sigma_2$ then, $r_1 \circ \Delta r_k \succ r_2 \circ \Delta r_k$ if

$$\frac{t-(\mu_1+\Delta\mu_k)}{\sigma_{1k}} < \frac{t-(\mu_2+\Delta\mu_k)}{\sigma_{2k}} \tag{6}$$

where $\sigma_{ik} = \left[\sigma_i^2 + \Delta\sigma_k^2\right]^{1/2}$, $i = 1,2$

That is

$$\Delta\mu_k > z_{12}(\Delta\sigma_k) = \frac{\sigma_{1k}(t-\mu_2) - \sigma_{2k}(t-\mu_1)}{\sigma_{1k} - \sigma_{2k}} \tag{7}$$

where $z_{12}(\Delta\sigma_k)$, the threshold between r_1 and r_2 when augmented by Δr_k is a function of only $\Delta\sigma_k$ of Δr_k.

Soroush [21] identifies six properties about $z_{12}(\Delta\sigma_k)$ and the domination between paths:

1. If $\mu_1 \ge \mu_2$, then $z_{12}(\Delta\sigma_k)$ is a decreasing function of $\Delta\sigma_k$.
2. If $\mu_1 < \mu_2$, then $z_{12}(\Delta\sigma_k)$ is an increasing function of $\Delta\sigma_k$.
3. If $\mu_2 < t < \mu_1$ or $\mu_1 > \mu_2 > t$, then $z_{12}(\Delta\sigma_k)$ is negative and decreases with $\Delta\sigma_k$.
4. If property 3 holds, then (7) is satisfied for any $\Delta\mu_k$ and $\Delta\sigma_k$, that is $r_1 \circ \Delta r_k \succ r_2 \circ \Delta r_k$ for any $\Delta\sigma_k$.
5. If property 3 does not hold (7) is satisfied for:
(5.1) $\mu_1 \ge \mu_2$, then $r_1 \circ \Delta r_k \succ r_2 \circ \Delta r_k$ for any $\Delta\sigma_k$.
(5.2) $\mu_1 < \mu_2$, then $r_1 \circ \Delta r_k \succ r_2 \circ \Delta r_k$ for some $\Delta\sigma_k$.

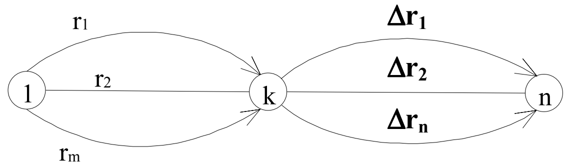

Figure 2. A PERT netwrok with internally disjoint paths between nodes.

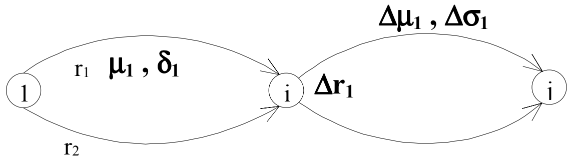

Figure 3. A PERT network with two paths and two path segments.

Remark 1 - We refer to the interval $\left[z_{12}(\Delta\sigma_k),+\infty\right]$, as the dominated interval of r_1, because it is internal if $\Delta\mu_k$ falls within it, then $r_1 \succ r_2$.

6. If property 3 and inequality (7) are not satisfied, then $r_2 \circ \Delta r_k \succ r_1 \circ \Delta r_k$ for some Δr_k.

Remark 2 - If $r_2 \circ \Delta r_k \succ r_1 \circ \Delta r_k$ for some Δr_k and the location of node v_n of the network is such that $\Delta^i\mu_k \in \left[0, z_{12}(\Delta\sigma_k)\right]$, where $\Delta^i\mu_k$ is the largest expected duration of a path segment between v_i and v_n, then there will be no path segment that would alter the relation $r_2 \circ \Delta r_k \succ r_1 \circ \Delta r_k$. Hence, r_1 can be eliminated.

Soroush's method to identify MCP is based on the above properties as a domination structure. Consider the network in Figure 2.

In Soroush's method, for any Δr_k, the domination between r_1 to r_m is identified and with known $\Delta\mu_k$, the appropriate path that dominates others is obtained. (This is named algorithm number 1 in Soroush's). Therefore, there are totally n evaluations as performed in Figure 2.

In Soroush's study, the domination evaluation between r_1 to r_m is performed, but this evaluation is not performed about path segments (Δr_k). Now consider the network in Figure 3.

Proposition 1 – Let $\sigma_1 < \sigma_2$, $\Delta\mu_1 \le \Delta\mu_2$, $\Delta\sigma_1 \le \Delta\sigma_2$, $\mu_1 > \mu_2 > t$ *or* $\mu_2 < t < \mu_1$, then in Figure 3, $r_1 \circ \Delta r_2$ is MCP if $\frac{t-\mu_2}{t-\mu_1} < \frac{\sigma_{21}-\sigma_{22}}{\sigma_{11}-\sigma_{12}}$.

Remark 3 - This proposition means that in the mentioned case, among four paths, $r_1 \circ \Delta r_2$ has smallest z.

Proof - Based on conditions in the proposition, $\sigma_{11}-\sigma_{21} < \sigma_{12}-\sigma_{22}$. Now, the condition in which $z_{12}(\Delta\sigma_1) > z_{12}(\Delta\sigma_2)$ would be:

$$\frac{\sigma_{11}(t-\mu_2)-\sigma_{21}(t-\mu_1)}{\sigma_{11}-\sigma_{21}} > \frac{\sigma_{12}(t-\mu_2)-\sigma_{22}(t-\mu_1)}{\sigma_{12}-\sigma_{22}} \tag{8}$$

Because the two denominators of the above relation are negative (Property 3), then we have

$$\frac{t-\mu_2}{t-\mu_1} < \frac{\sigma_{21}-\sigma_{22}}{\sigma_{11}-\sigma_{12}} \tag{9}$$

If in this proposition $\Delta\mu_1 > \Delta\mu_2$, then from property 3 we have

$$\Delta\mu_1 > \Delta\mu_2 > z_{12}(\Delta\sigma_1) > z_{12}(\Delta\sigma_2) \tag{10}$$

Hence, the domination of $r_1 \circ \Delta r_2$ is proved.

Generalization of Proposition 1 - Let $\Delta\sigma_1 < \Delta\sigma_2$, $\mu_1 > \mu_2 > t$ or $\mu_2 < t < \mu_1$ and $\sigma_1 \le \sigma_2$, then $r_1 \circ \Delta r_2$ is MCP at Figure 3, if $\frac{t-\mu_2}{t-\mu_1} < \frac{\sigma_{21}-\sigma_{22}}{\sigma_{11}-\sigma_{12}}$.

Remark 4 – (i) Let $\sigma_i < \sigma_j$ and $\Delta\sigma_1 < \Delta\sigma_2 < \ldots$, if $\mu_j < t < \mu_i$, then based on property 3 and generalization of proposition 1, the condition $\frac{t-\mu_j}{t-\mu_i} < \frac{\sigma_{jk}-\sigma_{jk'}}{\sigma_{ik}-\sigma_{ik'}}$ is satisfied for any $\underset{k<k'}{k,k'} \in \{1,\ldots,n\}$. In this case $\frac{t-\mu_j}{t-\mu_i}$ is negative and the right-hand-side of the relation is positive, then $\underset{m<k'}{r_i \circ \Delta r_m}$ is dominated, although in Soroush's domination structure only $r_j \circ \Delta r_k$ is dominated.

(ii) Let $\sigma_i < \sigma_j$, $\Delta\sigma_1 < \Delta\sigma_2 < \ldots$, if $\mu_i > \mu_j > t$, the condition of $\frac{t-\mu_j}{t-\mu_i} < \frac{\sigma_{jk}-\sigma_{jk'}}{\sigma_{ik}-\sigma_{ik'}}$, where $\underset{k<k'}{k,k'} \in \{1,\ldots,n\}$ should be checked and if satisfied, then $r_i \circ \Delta r_{k'}$ is MCP among four paths $r_i \circ \Delta r_k$, $r_i \circ \Delta r_{k'}$, $r_j \circ \Delta r_k$, $r_j \circ \Delta r_{k'}$. But based on Soroush's domination structure only $r_i \circ \Delta r_k$ dominates others. Soroush provides algorithm number 2 to identify MCP in networks as Figure 2 shows. Based on proposition 1, this algorithm is developed.

DEVELOPEMENT OF SOROUSH'S ALGORITHM NUMBER 2 TO IDENTIFY MCP IN NETWORK FIGURE 2

Step 1 - (Indexing and arranging)

(i) Index r_i, i = 1, ... , m, such that $\sigma_1 < \sigma_2 < \ldots < \sigma_m$.

(ii) Arrange $\mu_i, i = 1,\ldots,m$, and deadline t in ascending order.

$$\mu_{i_1} < \mu_{i_2} < \ldots < \mu_{i_j} < t < \mu_{i_{j+1}} < \ldots < \mu_{i_m} \tag{11}$$

(iii) Index $\Delta r_k, k = 1,\ldots,n$, such that $\Delta\sigma_1 > \Delta\sigma_2 > \ldots > \Delta\sigma_n$.

Step 2 – (i) Consider the paths, which satisfy the conditions $\mu_j < t < \mu_i$ and $\mu_i > \mu_j > t$.

(ii) Consider the paths, which satisfy the condition $\mu_j < t < \mu_i$, then proposition 1 is checked so that r_i would not be eliminated later.

Step 3 - Consider the indices $i_1 \ldots i_m$ given by (11) in turn and eliminate the dominated path r_{i_k} (Property 4) if its index i_k, $k \in \{1,\ldots,j\}$ is larger than some i_l that appear later in the order $i_{j+1} \ldots i_m$. Similarly, eliminate the dominated path r_{i_k} if its index $i_k, k \in \{j+1,\ldots,m\}$ is larger than some i_l that appear

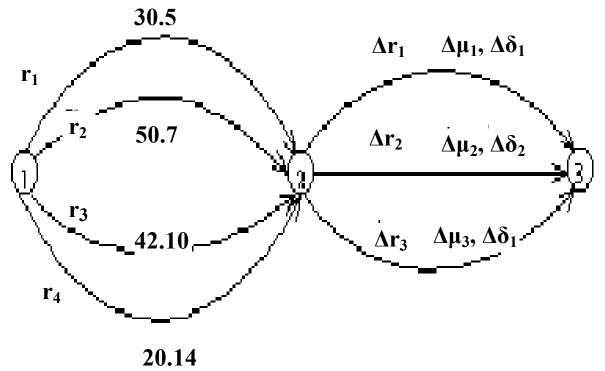

Figure 4. A PERT network. The two numbers beside each activity denote the expected duration and standard deviation of the activity.

later in the order $i_j+1 \ldots i_m$.

Step 4 – (i) If only r_m remains, terminate; $r_m \circ \Delta r_k, k = 1,\ldots,n$ (except those are dominated at Step 2) are the candidates for the MCP.

(ii) If more than one path remains, for any $\Delta r_k, k = 1,\ldots,n$ identify the path which dominates others (in Step 3, consider results of Step 2).

Step 5 - Calculate z for any candidate constructed in previous step. The smallest z identifies MCP.

In this algorithm only one condition $(\mu_j < t < \mu_i, \sigma_i < \sigma_j)$ of proposition 1 is used. If Step 2 is developed by merging other conditions $(\mu_i > \mu_j > t, \sigma_i < \sigma_j)$, better results are obtained.

In the following Example 1 is presented to demonstrate how the improved algorithm of Soroush works and reflects the difference in the manner of implementation of the new method and that on Soroush's.

Example 1 - Consider PERT network in Figure 4, Let $t = 35$ and $\Delta\sigma_1 = 8$, $\Delta\sigma_2 = 6$, $\Delta\sigma_3 = 3$, then $\mu_4 < \mu_1 < t < \mu_3 < \mu_2$ and based on Property 4, r_3 and r_4 are eliminated later. For domination about Δr_k, one case should be considered:
$\mu_4 < t < \mu_2, where \sigma_2 < \sigma_4, although \mu_1 < t < \mu_2$, $but \sigma_1 < \sigma_2, then$ proposition 1 does not hold.

Consider $\Delta r_1, \Delta r_2$ where $\Delta\sigma_1 > \Delta\sigma_2$. From proposition 2, $\frac{t-\mu_4}{t-\mu_2} < \frac{\sigma_{41}-\sigma_{42}}{\sigma_{21}-\sigma_{22}}$. Then $r_2 \circ \Delta r_2$ at Δr_2 stage evaluation and $r_2 \circ \Delta r_3$ at Δr_3 stage will be eliminated. After this evaluation, r_3 and r_4 are eliminated. Hence, for $\Delta r_k, k = 1,2,3$ the remainder paths are:

(i) $\Delta r_1; r_1 \circ \Delta r_1$ and $r_2 \circ \Delta r_1$ will be evaluated.

(ii) $\Delta r_2; r_2 \circ \Delta r_2$ remains (at Soroush's two paths $r_1 \circ \Delta r_2$ and $r_2 \circ \Delta r_2$ remain).

(iii) $\Delta r_3; r_1 \circ \Delta r_3$ remains (at Soroush's two paths $r_1 \circ \Delta r_3$ and $r_2 \circ \Delta r_3$ remain).

IDENTIFYING MCP IN GENERAL CASE

Based on the domination structure, a labeling algorithm for general case is presented.

Labeling Algorithm to Identify MCP in a PERT Network In this section a simple labeling algorithm based on the domination structure is derived that identifies MCP from the starting node to any node. Any label contains three elements. For instance the label of node v_i is written as; $(\mu_i, \Delta\sigma_i, j)$, where μ_i and σ_i are the mean and standard deviation of MCP to v_i and j is the number of a node that v_i has been labeled from it. Label of starting node is (0, 0, -). Node v_i should be labeled when all nodes connected to v_i have been labeled before.

Step 1 - Labeling starting node.

Step 2 - Use the CPM or a longest path algorithm to determine the largest mean duration $\Delta^i\mu$ of a path segment between each node v_i, i = 2, ... ,n.

Step 3 - (Labeling node v_i)

Consider the nodes i_1 to i_m which are connected to v_i. They are labeled before. So, we have m paths ending to node i. Then m nodes connected to v_i have labels as $(\mu_{i_k}, \sigma_{i_k}, j_{i_k})$, where k = 1,..., m and for m paths to v_i,

$$\mu_k = \mu_{i_k} + \mu'_k \quad \sigma_k = \left[\sigma_{i_k}^2 + \sigma_k^2\right]^{\frac{1}{2}} \quad , k = 1,\ldots,m \qquad (12)$$

where μ_k and σ_k are the mean duration and

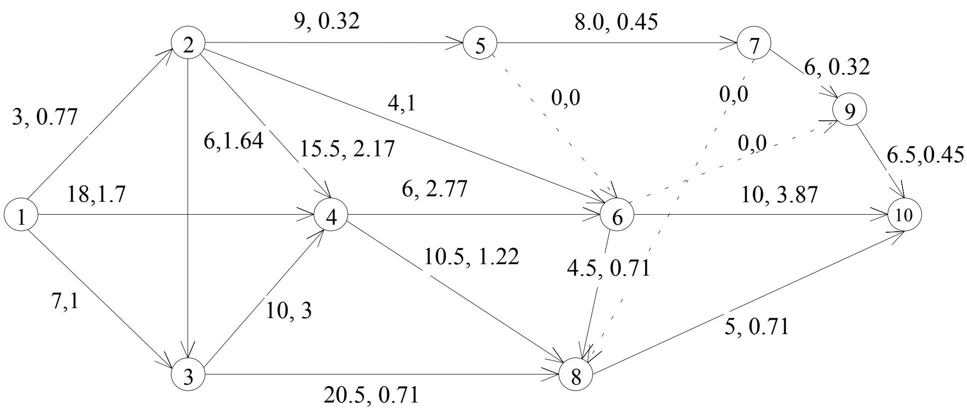

Figure 5. A PERT network [21]. The two numbers beside each activity denote the expected duration and standard deviation of the activity.

standard deviation of paths r_k to v_i respectively. Apply previous algorithm (Development of Soroush's algorithm).

For MCP identification to v_i, use $\Delta^i\mu$. If r_{i_k} dominates other paths, then label of v_i would be

$$\left(\mu_{i_k} + \mu_k,\ \left(\sigma_{i_k}^2 + \sigma_k^2\right)^{\frac{1}{2}}, i_k\right) \tag{13}$$

This procedure continues until v_n takes a label.

Remark 5 - From the third element of v_n label and backward movement from the ending node to the first node, MCP will be identified.

At this stage the algorithm is complete to find MCP. The following example is presented to illustrate the matter and to show how the algorithm works.

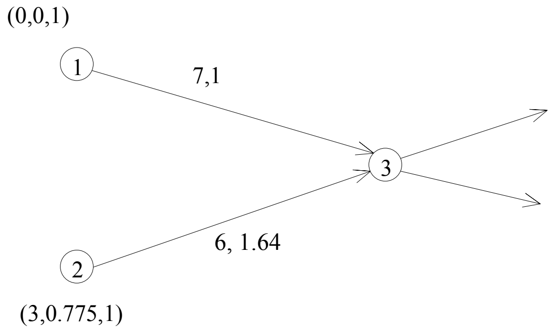

Figure 6. Labeling of node 3.

Example 2 - Consider PERT network in Figure 5;

$\Delta^2\mu = 31.5$ $\quad \Delta^3\mu = 26$ $\quad \Delta^4\mu = 16$ $\quad \Delta^5\mu = 20.5$

$\Delta^6\mu = 10$ $\quad \Delta^7\mu = 12.5$ $\quad \Delta^8\mu = 5$ $\quad \Delta^9\mu = 6.5$

Suppose the objective is MCP identification at t = 32. Label of v_2 is (3,0.775,1) because this node has only one entry of v_1. Identification of v_3 label:

$z_{12}(\Delta\sigma_1) = 26.074$, $\Delta^3\mu > 26.074$ and $\Delta^3\mu \in [0, 26.074]$

Based on Property 6, $r_2 \circ \Delta r_k$ dominates other paths for any $\Delta r_k, k = 1,2$ and then v_3 is labeled from v_2 as (9, 1.817,2). Labels for other nodes would be

$v_4 : (8.5, 2.302, 2), v_5 : (12, 0.832, 2), v_6 : (24.5, 3.605, 4),$
$v_7 : (20, 0.9495, 2), v_8 : (29.5, 1.95, 3), v_9 : (26, 1, 7),$
$v_{10} : (34.5, 2.07, 8).$

MCP at t = 32 moving backward from v_{10} is 1-2-3-8-10 and estimation of F(t) based on (4) is 0.1151.

The algorithm presented in this section recognizes the most critical path using the dominance structure. The authors of this paper provide the possibility of omitting of some other paths during the labeling process of algorithm (recognition of MCP). This possibility is made through the addition of some new conditions for the dominance structure constructed by Soroush. In fact, the domain of dominance structure is extended.

NEW ESTIMATION OF F(t) BASED ON MCP DEFINITION

To estimate F(t), relation (4) is written as:

$$p\{T^* \le t\} = p\{T_i \le t\} . p\{T_j \le t\} . \frac{p\{T_k \le t, T_i \le t, \forall k, k \ne i, j \mid T_j \le t\}}{p\{T_i \le t\}} \quad (14)$$

The ratio located in the right-hand-side of (14) may be greater or smaller than one. Then we have

$$p\{T^* \le t\} \lesseqgtr p\{T_i \le t\} . p\{T_j \le t\} \quad (15)$$

Based on MCP definition, r_i is the path that $p\{T_i \le t\}$ is smallest of all. Therefore;

$$p\{T_j \le t\} \in [p\{T_i \le t\}, 1] \quad (16)$$

The path $r_j \in R$ is selected such that $p\{T_i \le t\}\ p\{T_j \le t\}$ provides a new estimation of F(t).

Let $t_i > t_{i-1}$ and $\frac{MCP_i}{MCP_{i-1}} = \Delta_i$, where MCP_i is estimation of F(t) with MCP method at t_i. Also,

$$MCP_i^* = MCP_i . p\{T_j \le t\} \quad (17)$$

where MCP_i^* is a new estimation of F(t) at t_i. In order to preserve Δ_i amount as a maximum increase for estimation from t_{i-1} to t_i we should have:

$$\frac{MCP_i^*}{MCP_{i-1}^*} \le \Delta_i \quad (18)$$

Also $\frac{MCP_i^*}{MCP_{i-1}^*}$ is greater than one. Then;

$$1 < \frac{MCP_i^*}{MCP_{i-1}^*} \le \Delta_i \quad (19)$$

$$1 < \frac{MCP_i \cdot p\{T_j \le t\}}{MCP_{i-1}^*} \le \Delta_i \quad (20)$$

$$\frac{1}{a} < P\{T_j \le t\} \le \frac{\Delta}{a} \quad (21)$$

where $a_i = \frac{MCP_i}{MCP_{i-1}^*}$

Relation 2.1 determines a domain through which the j-th path can be selected. This path can be beneficial to construct a new estimate of F(t) with the help of MCP_i^*. But the main attention in the design of algorithm of this section is strictly the selection of j-th path in relation 2.1 in such a way that it would be in the nearest distance to upper bound; namely $\frac{\Delta_i}{a_i}$. In other words, it would be more desirable than the paths are selected for which $P\{T_j \le t\} \ge 0.9$.

Remark 6 - MCP* estimates F(t) and does not provide a bound for it.

Proof - Based on relation 2.1, we have

$$\frac{MCP_{i-1}^*}{MCP_i} < p\{T_j \le t\} < \frac{MCP_i . MCP_{i-1}^*}{MCP_{i-1} . MCP_i} \quad (22)$$

$$MCP_{i-1}^* < MCP_i . p\{T_j \le t\} < \frac{MCP_{i-1}^*}{MCP_{i-1}} . MCP_i \quad (23)$$

$$MCP_{i-1}^* < MCP_i^* < \frac{MCP_{i-1}^*}{MCP_{i-1}} . MCP_i \quad (24)$$

According to the definition $\frac{MCP_{i-1}^*}{MCP_{i-1}} < 1$ and the inequality of the left hand side, namely $MCP_{i-1}^* < MCP_i^*$, also holds true; according to the definition of probability distribution. So we have

$$MCP_i^* < bMCP_i \quad , \quad b < 1 \quad (25)$$

where $b = \frac{MCP_{i-1}^*}{MCP_{i-1}}$. We know that $F(t) < MCP_i$. But, in general, the kind of relation between $bMCP_i$ and F(t) is unknown. In other word, depending on b, F(t) may be less than or greater than $bMCP_i$ .So, F(t) is estimated by the appropriate determination of j-th path and $p\{T_j \le t\}$.

TABLE 1. Comparison of New Estimation and Soroush's to Approximate F(t) Approximation.

Problem No.	Network Size $\lvert V \rvert$	$\lvert A \rvert$	$\frac{SAE_N}{SAE_S}$
1	4	5	0.944
2	5	8	0.973
3	7	9	0.975
4	7	10	0.892
5	6	15	0.947
6	10	21	0.935
7	10	21	0.914

Approximation Algorithm for Estimating F(t)

Step 1 - Compute F(t) for all values of t; with the help of MCP and using algorithm explained in section 5.

Step 2 - Compute Δ_i for any t_i. For the first t, Δ_i should not be computed.

Step 3 - Consider the first estimation (MCP_i^* being equal to MCP_i) and for the next t_i recognize a path in the network by means of (21) and compute MCP_i^*. To find the appropriate path (for example r_j) to compute $p\{T_j \le t_i\}$and to gain MCP_i^*, the following instruction is presented.

The Instruction to Find the Appropriate Path to Determine $p\{T_j \le t_i\}$ for MCP_i^* Computation

Step 1 - Compute (21). (Show the resulting interval in the form of (a, b)).

Step 2 - If for all z_{kn}, k = 1,..., l, $p\{z \le z_{kn}\} > b$, finding a path that its probability at t_i locates in (a,b) interval would be impossible. Then $MCP_i^* = MCP_i$.

Step 3 - If for all z_{kn}, k = 1,..., l, $p\{z \le z_{kn}\} < a$, there is a chance to find a path that its probability comes in (a, b). Here, two strategies are proposed:
(i) Ignore to find a path whose probability comes in (a,b) and $MCP_i^* = MCP_i$.
(ii) Name the greatest z_{kn}, k = 1, ..., l as z_{hn} and the node before the last node of network that z_{hn} was computed via it, h. Other paths from h to the last node (if exist) give probability more than z_{hn}. If appropriate path is found, use it. Otherwise; $MCP_i^* = MCP_i$ or take another z_{hn} which is at the nearest distance to z_{hn} and repeat this step.

Step 4 - If for all z_{kn}, k = 1, ..., l one or some r_j are found whose probability come in (a, b) two strategies are proposed:

(i) Select the greatest z_{kn} and compute MCP_i^* .

(ii) After the greatest zkn (i. e. z_{hn}) is found, paths which are joined to node h should be recognized and their z should be computed and if they come at (a,b) the most appropriate path should be chosen.

NUMERICAL ANALYSIS

In this section some problems to evaluate new estimation of F(t) are presented. Totally seven-problems are defined as;

(i) Two problems of Dodin's[5] are selected. In the first problem each activity is exponentially distributed with $\lambda = 1.5$ and in the second problem each activity has the realizations 1,2,3,4,5 each with probability 0.2.

(ii) One problem of Ringer's [2] is selected and F(t) is calculated analytically. Each activity is distributed normally or exponentially.

(iii) Four problems are selected from Fatemi Ghomi's work [22], that all activities are defined by discrete random variables. In these problems F(t) are computed by means of simulation. Table 1 shows the results for the above-mentioned problems. Column 4 shows $\frac{SAE_N}{SAE_S}$, where SAE_N is the sum of absolute errors of new estimation of F(t) and SAE_S is the sum of absolute errors of Soroush's estimation of F(t) at a given time.

The results indicate that the new method provides better estimation than that of Soroush's. Meanwhile, the error ratio does not prevail a significant relation between the magnitude of the error and network size. The major factor in error reduction corresponding to the new method is that it uses two paths to estimate F(t). The initial estimate of the new method is made with the help of Soroush's algorithm. Based on this reason and (21), column 4 represents numbers completely close to one. For those networks having large number of paths, the new method has the increased chance to obtain good results. Soroush's method provides an upper bound for F(t), but the new method does not have this property. In four problems of Table 1, for some amount of time, the new method provides smaller estimation for F(t). Usually at small amount of time, a large amount for Δ is obtained, then (21) represents a large interval. In this case, it is probable to obtain a path to compute MCP*. When the initial estimate of new method has the value between 0.9 to 1 for F(t), if a path is found at (21), small variation occurs in the recent computation. This indicates that the new method has suitable performance. Also, when F(t) has a small amount, by obtaining large Δ, the possibility to find appropriate path at (21) increases. These points were led to derive (21).

CONCLUSION

In this paper, the purpose was MCP identification and to estimate completion time distribution function of a PERT network. A labeling algorithm is derived that acts as a simple mechanism to identify MCP from the starting node to any node in the network. An estimation method for direct computation of F(t) is presented that gives satisfactory answers in comparison with those of Soroush's.

REFERENCES

1. Harthley, H. O. and Wortham, A. W., "A Statistical Theory for PERT Critical Path Analysis", ***Management Science***, Vol. 12, No.10, (1966), 469-481.
2. Ringer, L. J., "Numerical Operators for Statistical PERT Critical Path Analysis", ***Management Science***, Vol. 16, No. 2, (1969), 136-143.
3. Kulkarni, V. G. and Adlakha, V. G., "Markov and Markov Regenerative PERT Network", ***Journal of Operations Research***, Vol.34, No. 5, (1980), 769-781.
4. Martin, J. J., "Distribution of the Time Through a Directed, Acyclic Network", ***Operations Research***, Vol. 13, No.1, (1965), 46-66.
5. Dodin, B. M., "Bounding the Project Completion Time Distribution in PERT Network", ***Journal of Operations Research***, Vol. 33, No. 4, (1985), 826-881.
6. Vobill, P. and Trahan, M., "The Completion Time of PERT Network", ***Journal of Operations Research***, Vol. 25, No. 1, (1977), 15-29.
7. Kleindorfer, G. B., "Bounding Distributions for a Stochastic Acyclic Network", ***Journal of Operations Research***, Vol. 19, No. 7, (1977), 1586 - 1601.
8. Van Slyke, R. M., "Monte Carlo Methods and the PERT Problem" ***Operations Research***, Vol. 11, No. 5, (1963), 839-860.
9. Sculli, D., "The Completion Times of PERT Network", ***Journal of the Operational Research Society***, Vol. 34, No 2, (1983), 155-158.
10. Kamburowski, J., "An Upper Bound on the Expected Completion Time of PERT Network", ***European Journal of Operational Research Society***, Vol. 21, No. 2, (1985), 206-212.
11. Fulkerson, D. R., "Expected Critical Path Length in PERT Network", ***Operations Research***, Vol. 10, No. 6, (1962), 808-817.
12. Clingen, C. T., "A Modification of Fulkerson's PERT Algorithm", ***Operations Research***, Vol. 12, No.4, (1964), 629-632.
13. Elmaghraby, S. E., "On the Expected Duration of PERT Type Network", ***Management Science***, Vol. 13, No.5, (1967), 299-306.
14. Robillard, P. and Trahan, M., "Expected Completion in Time.in Part Network", ***Operation Research***, Vol. 24, No. 1, (1976), 177-182.
15. Lindsey, J. H., "An Estimate of Expected Critical Path Length in PERT Network", ***Journal of Operations Research***, Vol. 20, No.4, (1972), 800-812.
16. Dodin, B. M., "Approximating the Distribution Function in Stochastic Network", ***Computers and Operations Research***, Vol. 12, No.3, (1985), 251-264.
17. Sigal, C. E., Pritsker, A. A. B. and Solberg, J. J., "The Use of Cutsets in Monte Carlo Analysis of Stochastic Network", ***Mathematics and Computers in Simulation***, Vol. 21, (1979), 376-384.
18. Fisher, D. L., Saisi, D. and Goldstein, W. M., "Stochastic PERT Networks: OP Diagrams, Critical Paths and the Project Completion Time", ***Computers and Operations Research***, Vol. 12, No.5, (1985), 471-482.
19. Dodin, B. and Elmaghraby, S. E., "Approximating the Criticality Indices of the Activities in PERT Networks", ***Management Science***, Vol. 31, No. 2, (1985), 207-223.
20. Dodin, B. M., "Determining the K Most Critical Paths in PERT Network", ***Operations Research***, Vol. 32, No. 4, (1984), 859-877.
21. Soroush, H. M., " The Most Critical Path in a PERT Network", ***Journal of the Operational Research***

Society, Vol. 45, No. 3, (1994), 287-300.
22. Fatemi Ghomi, S. M. T., Hashemin, S. S., Teimouri, E., and Jolai Ghazvini, F., “Developing the Approaches to Compute Completion Time Distribution Function for Stochastic PERT Networks”, ***Research work***, Amirkabir University of Technology, Tehran, Iran (1994).