FREEZING IN A FINITE SLAB USING EXTENSIVE PERTURBATION EXPANSIONS METHOD

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Abstract In this paper Mathematica is used to solve the moving boundary problem of freezing in a finite slab for higher order perturbations. Mathematica is a new system which makes it possible to do algebra with computer. More specifically, it enables researchers to find the location of the ice at any time for as high order of perturbation as one whishes. Using of Mathematica and outer solution and an inner solution for the quantities involved are developed and then these solutions are matched together using the method of matched asymptotic expansions. At the end the composite solutions are compared to the available low-order solutions.

Key Words Moving Boundary, Transient Conduction, Freezing Problem, Perturbation, Finite Slab

چکیده دراین مقاله Mathematica را برای مساله با مرزهای متحرک انجماد روی یک صفحه صاف و برای مرتبه های بالای اختلال به کار می گیریم. Mathematica یک سیستم جدید است و امکان انجام دادن عملیات جبری را با استفاده از کامپیوتر فراهم می کند. بنابراین محل انجماد در هر لحظه را تا هر مرتبه ای از اختلال که موردنظر باشد می توان بدست آورد. دراین جا با استفاده از Mathematica یک حل داخلی و یک حل خارجی برای کمیات درگیر در مساله بدست آورده و سپس این حل ها را توسط روش انطباقی مجانبی به یکدیگر پیوند می دهیم. در خاتمه حل های مرکب با حل های از مرتبه پائین در دسترس، مقایسه می شوند.

INTRODUCTION

The problem of freezing in a finite slab using perturbation methods has been solved by Weinbaum and Jiji [1], and Aziz [2] [2]. These works, however, only consider the first two perturbation terms in which the outer solution and inner solution of some quantities turn out to be the same. Therefore, the asymptotic matching plays no role and the boundary layer (inner region) in this singular problem does not show itself. This lack of information, among others, can be compensated by solving the problem for higher orders of perturbation quantities. The compensation procedure, nevertheless, involves a large amount of algebra which is overwhelming and impossible if the order is rather high. This difficulty can be overcome by using Mathematica developed recently by Wolfram [3]. Mathematica is a system for doing mathematics by computer; therefore, it can be very useful in solving problems which uses perturbation techniques. Ideally, Mathematica can be employed to solve problems using perturbation techniques with very high orders of accuracy, say exact.

In this paper the outer and inner solutions for the problem of freezing in a finite slab with very high order of accuracy are considered. Then these solutions are matched together asymptotically to find the composite solution. Since the presentation of results for higher order terms needs large amount of paper, here we only present three perturbation terms of the resulting equations.

In the next section, the governing equations and the perturbation techniques are presented.

FORMULATION OF THE PROBLEM

This problem is concerned with a situation where the difficulty is due to the presence of a moving boundary whose position is not known a priori. A relatively simple moving boundary problem is the inward freezing of a finite slab shown in Figure 1.

Initially the liquid assumed to be at its freezing temperature T_f. At time $t>0$, the face at $x=0$ is maintained at constant subfreezing temperature T_a, so that $T_a<T_f$.

As heat is extracted from the liquid, it begins to freeze. Let the freezing front, at any instant of time, be located at distance x_f. It is assumed that the temperature of unfrozen liquid changes throughout the process. For the solid phase and liquid phase, the applicable equation is that of one-dimensional transient conduction which may be written as

$$\alpha_s \frac{\partial^2 T_s}{\partial x^2} = \frac{\partial T_s}{\partial t} \qquad (1)$$

$$\alpha_l \frac{\partial^2 T_l}{\partial x^2} = \frac{\partial T_l}{\partial t} \qquad (2)$$

where α is the thermal diffusivity and subscripts s and l denote solid and liquid phases, respectively (Arpaci [4]). The boundary conditions for the two phases are

$$T_s(0,t) = T_a \qquad (3)$$

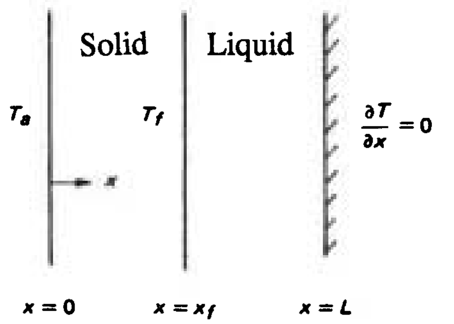

Figure 1. Freezing in a finite slab.

$$T_s[x_f(t),t] = T_L[x_f(t),t] = T_f \qquad (4)$$

$$\frac{\partial T_l(L,t)}{\partial x} = 0 \qquad (5)$$

The initial conditions on liquid temperature T_l and interface location $x_f(t)$ are

$$T_l(x,0) = T_i \qquad (6)$$

$$x_f(0) = 0 \qquad (7)$$

The energy balance at the interface requires

$$k_s \frac{\partial T_s(x_f,t)}{\partial x} - k_l \frac{\partial T_l(x_f,t)}{\partial x} = \rho H \frac{dx_f}{dt} \qquad (8)$$

By introducing the dimensionless quantities

$$\theta = \frac{k_s(T_s - T_f)}{k_l(T_i - T_f)} \qquad \phi = \frac{T_l - T_f}{T_i - T_f} \qquad \xi = \frac{x}{x_f(t)}$$

$$\eta = \frac{x - x_f}{L - x_f} \qquad \tau = \frac{\varepsilon \alpha_l t}{L^2} \qquad \sigma = \frac{x_f}{L} \qquad \varepsilon = \frac{c(T_i - T_f)}{H} \qquad (9)$$

Equations 1-7 become

$$\frac{\partial^2 \theta}{\partial \xi^2} = \varepsilon\left(\frac{\alpha_l}{\alpha_s}\right)\left(\sigma^2 \frac{\partial \theta}{\partial \tau} - \xi \frac{d\sigma}{d\tau}\frac{\partial \theta}{\partial \xi}\right) \qquad (10)$$

$$\frac{\partial^2 \phi}{\partial \eta^2} = \varepsilon\left[(1-\sigma)^2 \frac{\partial \phi}{\partial \tau} - (1-\sigma)(1-\eta)\frac{d\sigma}{d\tau}\frac{\partial \phi}{\partial \eta}\right] \qquad (11)$$

subject to the boundary conditions

$$\theta(0,\tau) = \frac{k_s(T_a - T_f)}{k_l(T_i - T_f)} = \theta_a \qquad (12)$$

$$\theta(l,\tau) = \phi(0,\tau) = 0 \qquad (13)$$

$$\frac{\partial \phi(l,\tau)}{\partial \eta} = 0 \qquad (14)$$

and the initial conditions

$$\phi(\eta, 0) = 1 \tag{15}$$

$$\sigma(0) = 0 \tag{16}$$

The energy balance at the interface, Equation 8 becomes

$$(1-\sigma)\frac{\partial\theta(l,\tau)}{\partial\xi} - \partial\frac{\partial\phi(0,\tau)}{\partial\eta} = \sigma(1-\sigma)\frac{d\sigma}{d\tau} \tag{17}$$

Mathematically, we seek solutions to Equations 10 and 11, subject to the boundary and initial conditions, Relations 12-16 and the interfacial energy balance, Relation 17, for a given value of θ_a. The ratio of the diffusivities α_l/α_s is of course known. Here our choice of perturbation quantity is the parameter

$$\varepsilon = \frac{c(T_i - T_f)}{H}$$

which in phase change literature is called the *Stefan number*. It represents the ratio of the sensible heat to the latent heat stored during the phase change process. The magnitude of *Stefan number* ε can vary considerably depending on the material and the temperature differences involved. For water and paraffin waxes, $\varepsilon<1$, for metals $1<\varepsilon<10$, and for materials such as silicates ε reaches up to several hundreds. Thus, for substances such as water and for materials used in latent heat thermal energy storage devices a perturbation analysis based on the assumption of small ε is appropriate.

Since the parameter ε appears in front of the highest order of derivatives of θ_0 and ϕ with respect to τ, the initial conditions 15 and 16 cannot be satisfied, Nayfeh [5] and Bender & Orszag [6]. An inner expansion near $\tau = 0$ will have to be introduced in addition to the regular perturbation.

SOLUTION OF THE PROBLEM

In this section are considered the outer and inner solution of the governing equations along with their appropriate boundary conditions. To find the outer solution, an outer expansion of the form (outside the inner region) is used

$$\theta(\xi, \tau, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{n/2}\theta_n(\xi, \tau) \tag{18}$$

$$\phi(\eta, \tau, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{n/2}\phi_n(\eta, \tau) \tag{19}$$

$$\sigma(\tau, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{n/2}\sigma_n(\eta, \tau) \tag{20}$$

which upon substitution into Equations 10, 11 and 17 gives (Only three terms are presented here)

$$\varepsilon^0: \frac{\partial^2\theta_o}{\partial\xi^2} = 0 \tag{21}$$

$$\frac{\partial^2\phi_o}{\partial\eta^2} = 0 \tag{22}$$

$$(1-\sigma_0)\frac{\partial\theta_0(1,\tau)}{\partial\xi} - \sigma_0\frac{\partial\phi_0(0,\tau)}{\partial\eta} = \sigma_0(1-\sigma_0)\frac{d\sigma_0}{d\tau} \tag{23}$$

$$\varepsilon^{1/2}: \frac{\partial^2\theta_1}{\partial\xi^2} = 0 \tag{24}$$

$$\frac{\partial^2\phi_1}{\partial\eta^2} = 0 \tag{25}$$

$$(1-\sigma_0)\frac{\partial\theta_1(1,\tau)}{\partial\xi} - \sigma_1\frac{\partial\theta_0(1,\tau)}{\partial\xi} - \sigma_0\frac{\partial\phi_1(0,\tau)}{\partial\eta} - \sigma_1\frac{\partial\phi_0(0,\tau)}{\partial\eta} = \sigma_0(1-\sigma_0)\frac{d\sigma_1}{d\tau} - \sigma_0\sigma_1\frac{d\sigma_0}{d\tau} + \sigma_1(1-\sigma_0)\frac{d\sigma_0}{d\tau} \tag{26}$$

$$\varepsilon: \frac{\partial^2\theta_2}{\partial\xi^2} = \left(\frac{\alpha_1}{\partial_s}\right)\left(\sigma_0^2\frac{\partial\theta_0}{\partial\tau} - \xi\sigma_0\frac{d\sigma_0}{d\tau}\frac{\partial\phi_0}{\partial\eta}\right) \tag{27}$$

$$\frac{\partial^2\phi_2}{\partial\eta^2} = (1-\sigma_0)\frac{\partial\phi_0}{\partial\tau} - (1-\sigma_0)(1-\eta)\frac{d\sigma_0}{d\tau}\frac{\partial\phi_0}{\partial\eta} \tag{28}$$

$$(1-\sigma_0)\frac{\partial\theta_2(1,\tau)}{\partial\xi}-\sigma_1\frac{\partial\theta_1(1,\tau)}{\partial\xi}-\sigma_2\frac{\partial\theta_0(1,\tau)}{\partial\xi}-\sigma_0\frac{\partial\phi_2(0,\tau)}{\partial\tau}$$
$$-\sigma_1\frac{\partial\phi_1(0,\tau)}{\partial\eta}-\sigma_2\frac{\partial\phi_0(0,\tau)}{\partial\eta}=-\sigma_0\sigma_2\frac{d\sigma_0}{d\tau}-\sigma_0\sigma_1\frac{d\sigma_1}{d\tau}+$$
$$\sigma_0(1-\sigma_0)\frac{d\sigma_2}{d\tau}-\sigma_1^2\frac{d\sigma_0}{d\tau}+\sigma_1(1-\sigma_0)\frac{d\sigma_1}{d\tau}-\sigma_2(1-\sigma_0)\frac{d\sigma_0}{d\tau} \tag{29}$$

The solution of Equations 21-29 can be obtained as

$$\theta_0=\theta_a(1-\xi) \tag{30}$$

$$\theta_1=0 \tag{31}$$

$$\theta_2(\xi,\tau)=\frac{1}{2}\alpha\,\theta_a(\xi-\xi^2)\sqrt{c_0-2\theta_a\tau}+\frac{1}{6}\alpha\,\theta_a^2(\xi-\xi^3) \tag{32}$$

$$\phi_0=0 \tag{33}$$

$$\phi_1=0 \tag{34}$$

$$\phi_2(\eta,\tau)=0 \tag{35}$$

$$\sigma_0\,(c_0-2\theta_a\tau)^{1/2} \tag{36}$$

$$\sigma_1=0 \tag{37}$$

$$\partial_2=[-\frac{1}{3}\alpha\,\theta_a^2\,\tau+\frac{1}{6}\alpha(c_0-2\theta_a\tau)^{3/2}-\frac{1}{6}\alpha\,c_0\sqrt{c_0}]\,/\sqrt{c_0-2\theta_a\tau} \tag{38}$$

Here c_0 is a constant which can be determined from matching the inner and outer solutions. The outer expansions are therefore

$$\theta(\xi,\tau,\varepsilon)=\theta_a 1-\xi)+\frac{1}{2}\alpha\,\theta_a\,\xi(1-\xi)+\ldots \tag{39}$$

$$\theta(\eta,\tau,\varepsilon)=0+O(\varepsilon^n) \tag{40}$$

$$\sigma(\tau,\varepsilon)=(c_0-2\theta_a\tau)^{1/2}+\varepsilon\frac{-(1/3)\,\alpha\theta_a^2\,\tau+(1/6)\,\alpha(c_0-2\theta_a\tau)^{3/2}-1/6\,\alpha c_0\sqrt{c_0}}{\sqrt{c_0-2\theta_a\tau}}+\ldots \tag{41}$$

For the inner expansions, the following inner variables are introduced

$$T=\frac{\tau}{\varepsilon}$$
$$\hat{\theta}(\xi,T)=\theta(\xi,\tau) \tag{42}$$
$$\hat{\phi}(\eta,T)=\phi(\eta,\tau)$$
$$\hat{\sigma}(T)=\sigma(\tau)$$

and substituted into Equations 10, 11, and 17 give:

$$\frac{\partial^2\hat{\theta}}{\partial\xi^2}=\frac{\alpha_l}{\alpha_s}\left(\hat{\sigma}^2\frac{\partial\hat{\theta}}{\partial T}-\xi\hat{\sigma}\frac{d\hat{\sigma}}{dT}\frac{\partial\hat{\theta}}{\partial\xi}\right) \tag{43}$$

$$\frac{\partial^2\hat{\phi}}{\partial\eta^2}=(1-\hat{\sigma})^2\frac{\partial\hat{\phi}}{\partial T}-(1-\hat{\sigma})(1-\eta)\frac{d\hat{\sigma}}{dT}\frac{\partial\hat{\phi}}{\partial\eta} \tag{44}$$

$$\varepsilon\left[(1-\hat{\sigma})\frac{\partial\hat{\theta}(1,T)}{\partial\xi}-\hat{\sigma}\frac{\partial\hat{\phi}(0,T)}{\partial\eta}\right]=\hat{\sigma}(1-\hat{\sigma})\frac{d\hat{\sigma}}{dT} \tag{45}$$

By expanding $\hat{\theta}$, $\hat{\phi}$ and $\hat{\sigma}$ in terms of inner variables,

$$\hat{\theta}(\xi,T,\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^{n/2}\,\hat{\theta}_n(\xi,T) \tag{46}$$

$$\hat{\phi}(\eta,T,\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^{n/2}\,\hat{\phi}_n(\eta,T) \tag{47}$$

$$\hat{\sigma}(T)=\sum_{n=0}^{\infty}\varepsilon^{n/2}\hat{\sigma}_n(T) \tag{48}$$

and substituting into Equations 43-45 and presenting here only three terms of them, lead to

$$\varepsilon^0:\ \frac{\partial^2\hat{\theta}_o}{\partial\xi^2}=0 \tag{49}$$

$$\frac{\partial^2\hat{\phi}_o}{\partial\eta^2}=\frac{\partial\phi_0}{\partial T} \tag{50}$$

$$\hat{\sigma}_0=0 \tag{51}$$

$$\varepsilon^{1/2}:\ \frac{\partial^2\hat{\theta}_o}{\partial\xi^2}=0 \tag{52}$$

$$\frac{\partial^2\hat{\phi}_1}{\partial\eta^2}=\frac{\partial\hat{\phi}_1}{\partial T}-2\hat{\sigma}_1\frac{\partial\hat{\phi}_0}{\partial T}-(1-\eta)\frac{d\hat{\sigma}_1}{dT}\frac{\partial\hat{\phi}_0}{\partial\eta} \tag{53}$$

$$\frac{\partial^2 \hat{\theta}_0(1,T)}{\partial \xi} = \hat{\sigma}_1 \frac{d\hat{\sigma}_1}{dT} \tag{54}$$

$$\varepsilon : \frac{\partial^2 \hat{\theta}_2}{\partial \xi^2} = \alpha(\hat{\sigma}_1^2 \frac{\partial \hat{\theta}_0}{\partial T} - \xi \hat{\sigma}_1 \frac{d\hat{\sigma}_1}{dT}\frac{\partial \hat{\theta}_0}{\partial \xi}) \tag{55}$$

$$\frac{\partial^2 \hat{\phi}_2}{\partial \eta^2} = \frac{\partial \hat{\phi}_2}{\partial T} + \hat{\sigma}_1^2 \frac{\partial \hat{\phi}_0}{\partial T} - 2\hat{\sigma}_1 \frac{\partial \hat{\phi}_1}{\partial T} - 2\hat{\sigma}_2 \frac{\partial \hat{\phi}_0}{\partial T}$$

$$-(1-\eta)\frac{d\hat{\sigma}_1}{dT}\frac{\partial \hat{\phi}_1}{\partial \eta} - (1-\eta)\frac{d\hat{\sigma}_2}{dT}\frac{\partial \hat{\phi}_0}{\partial \eta} - (1-\eta)\hat{\sigma}_1 \frac{d\hat{\sigma}_1}{dT}\frac{\partial \hat{\phi}_0}{\partial \eta} \tag{56}$$

$$\hat{\sigma}_2 \frac{d\hat{\sigma}_1}{dT} - \hat{\sigma}_1 \frac{d\hat{\sigma}_1}{dT} + \hat{\sigma}_1 \frac{d\hat{\sigma}_2}{dT} = \frac{\partial \hat{\theta}_1(1,T)}{\partial \xi} - \hat{\sigma}_1 \frac{\partial \hat{\theta}_0(1,T)}{\partial \xi}$$

$$- \hat{\sigma}_1 \frac{\partial \hat{\phi}_0(0,T)}{\partial \eta} \tag{57}$$

The solutions of Equations 49-57 are

$$\hat{\theta}_0 = \theta_a(1-\xi) \tag{58}$$

$$\hat{\theta}_1 = 0 \tag{59}$$

$$\hat{\theta}_2(\xi,T) = -\frac{1}{6}\alpha\theta_a^2\xi^3 + \theta_a(\frac{1}{6}\alpha\theta_a - 1)\xi + \theta_a \tag{60}$$

$$\hat{\sigma}_0 = 0 \tag{61}$$

$$\hat{\sigma}_1 = (-2\theta_a T)^{1/2} \tag{62}$$

$$\hat{\sigma}_2 = \sum_{j=0}^{\infty} \left\{ \frac{2}{[(j+\frac{1}{2})\pi]^2} \exp\left\{ -\left[\left(j+\frac{1}{2}\right)\pi\right]^2 T \right\} - \frac{\sqrt{\pi}\,\mathrm{erf}\,[(j+\frac{1}{2})\pi\sqrt{T}]}{[(j+\frac{1}{2})\pi]^3\sqrt{T}} \right\} \tag{63}$$

$$\hat{\phi}_0 = \sum_{j=0}^{\infty} \left[\frac{2}{(j+\frac{1}{2})\pi} \sin\left(j+\frac{1}{2}\right)\pi\eta \right] \exp\left\{ -\left[\left(j+\frac{1}{2}\right)\pi\right]^2 T \right\} \tag{64}$$

Here, the other orders of $\hat{\phi}(\eta,T)$ are not presented due to their complexity. Equations 58-64 represent the inner solution of the three quantities θ, ϕ, and σ. Now match these solutions can be matched by using Van Dyke matching princiles, see Nayfeh [5] and Bender & Orszag [6]. The Composite solutions are found from the matching condition (for example for σ)

$$\sigma^{(c)} = \sigma^{(o)} + \sigma^{(i)} - (\sigma^{(i)})^{(o)} \tag{65}$$

in which $(\sigma^{(i)})^{(o)}$ is the limit of inner solution in terms of outer solution. Doing this the value $c_o = 0$ is obtained. Therefore, the composite solutions are

$$\theta^{(c)}(\xi,\tau) = \theta_a(1-\xi) + \varepsilon[\frac{1}{2}\alpha\theta_a(-2\theta_a\tau)^{1/2}(\xi-\xi^2) + \frac{1}{6}\alpha\theta_a^2(\xi-\xi^3)] \tag{66}$$

$$\sigma^{(c)} = (-2\theta_a\tau)^{1/2} + \varepsilon\left[\frac{(1/6)\,\alpha(-2\theta_a\tau)^{3/2} - (1/3)\,\alpha\theta_a^2\tau}{(-2\theta_a\tau)^{1/2}} + \sum_{j=0}^{\infty}\left(\frac{2}{[(j+\frac{1}{2})\pi]^2} \exp\left\{-\left[\left(j+\frac{1}{2}\right)\pi\right]^2 T\right\} - \frac{\sqrt{\pi}\,erf\,[(j+\frac{1}{2})\pi\sqrt{T}\,]}{[(j+\frac{1}{2})\pi]^3\sqrt{T}} \right) \right] + \ldots \tag{67}$$

and ϕ is not presented here due to its complexity.

As it can be seen from Equation 66, the composite solution is given in a function of time. This is seen from order ε on. Also it is seen that the outer and inner solutions differ and therefore the matching principle can be applied. As mentioned before the result of Reference 1 is not a function of time. Nor does it show the singularity in terms of time variable. Therefore, when the result of Equation 66 is drawn, the trend of an outer solution intersecting an inner solution can be observed, which indicates and displays the removal of singularity in terms of time. In the case of Equation 67 the correction of ice location in terms of time is presented.

SOME NUMERICAL RESULTS

Here some numerical results are presented. For example in Figure 2 a three-term expansion for the composite solution of temperature is presented along with the result of two-term expansion of this quantity which is not a function of time and has been done by Weinbaum and Jiji. As one can see from this figure, the time dependency of this quantity shows itself from the third term on (see Equation 66). The important point here is the existance of singularity with respect to time. Here the composite solution has a trend of an outer solution intersecting an inner solution, which is the indication of this singularity (time boundary layer). In Figure 3, a six-term solution of this quantity is presented. For example at $\tau = 0$, and $\xi = 0.5$, σ has the form of

$$\sigma = 0.5 + 0.0625\varepsilon + 0.01\varepsilon^2 + 0.0154\varepsilon^3 + 0.0243\varepsilon^4 + 0.0471\varepsilon^5 + O(\varepsilon^6)$$

Here the composite solution shows a more clear trend of an outer intersecting an inner solution. Figure 4,

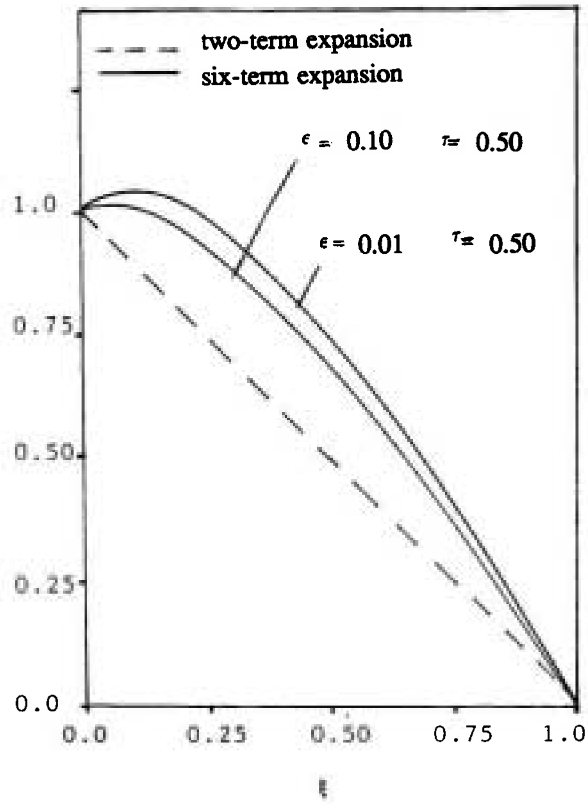

Figure 3. θ in terms of ξ.

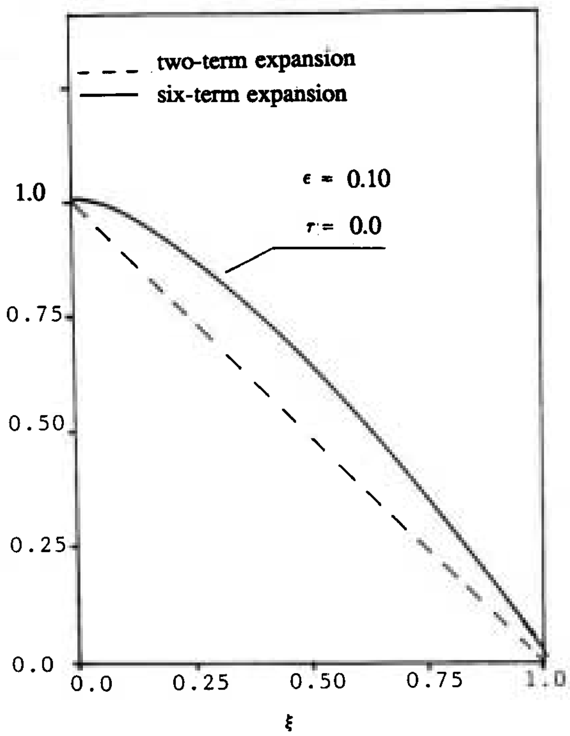

Figure 2. θ in terms of ξ.

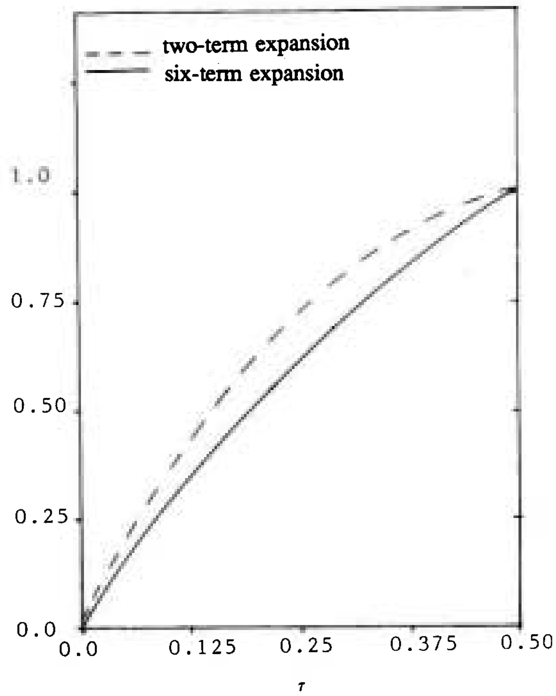

Figure 4. Ice front in terms of time, ε= 0.1.

shows the non-dimensional ice front in terms of time. The curve representing the six-term expansion is a correction to the two-term expansion.

The comparison of the results obtained here with lower-order results proves the necessity of solving this problem for higher orders. It is obvious that for liquids with ε>1, the technique of perturbation can not be applied and perhaps a numerical method can be employed which provides a lesser physical understanding.

LIST OF SYMBOLS

c	specific heat
H	latent heat
k	thermal conductivity
L	location of wall
t	time
T	temperature
x	cartesian coordinate system

Greek Letters

α	thermal diffusivity
θ	dimensionless temperature
, ξ	dimensionless variables
μ	mass density
ϕ	dimensionless temperature
τ	dimensionless time
σ	ice location
ε	perturbation parameter

Subscripts

s	solid phase
l	liquid phase
a	ambient
o	zeroth-order perturbation
1	first-order perturbation
2	second-order perturbation
f	ice front

Superscripts

i	inner quantity
i	inner solution
o	outer solution

REFERENCES

1. A. S. Weinbaum, and L. M. Jiji, "Conduction in Irregular Domains", *Journal of International Heat and Mass Transfer*, Vol. 20, (1977).

2. A. Aziz and T. Y. Na, "Perturbation Methods in Heat Transfer", Hermisphere Publishing Corporation, U.S.A., (1984).

3. S. Wolfram, "Mathematica a System for Doing Mathematics by Computers", Addison-Wesley Company, Inc. U.S.A., (1991).

4. V. S. Arpaci, "Conduction Heat Transfer", Addision-Wesley Publishing Company, Reading, Massachusetts, (1984).

5. A. H. Nayfeh, "Perturbation Methods", John Wiley & Sons, New York, U.S.A., (1992).

6. C. M. Bender and S. A. Orszag, "Advanced Mathematical Methods for Scientists and Engineers", McGraw-Hill Book Company, U.S.A., (1980).