OPTIMAL DESIGN OF GEOMETRICALLY NONLINEAR STRUCTURES UNDER A STABILITY CONSTRAINT

S. Pezeshk

Department of Civil Engineering
Memphis State University
Memphis, Tennessee, U. S. A.

Abstract This paper suggests an optimization-based methodology for the design of minimum weight structures with kinematic nonlinear behavior. Attention is focused on three-dimensional reticulated structures idealized with beam elements under proportional static loadings. The algorithm used for optimization is based on a classical optimality criterion approach using an active-set strategy for extreme limit constraints on the design variables. A fixed-point iteration algorithm based on the criterion that at optimum the nonlinear strain energy is equal for all members is used. Several examples are given to evaluate the validity of the underlying assumptions and to demonstrate some of the characteristics of the proposed procedure.

Key Words Optimal design, Optimality Criteria, Stability Constraints

چکیده این مقاله یک روش بهینه سازی را برای طراحی سازه های با حداقل وزن با رفتار سینتیکی غیر خطی پیشنهاد میکند. توجه بیشتر روی سازه های سه بعدی اسکلتی تحت بارهای تدریجی و ساکن تشکیل شده اند متمرکز شده است. الگاریتم مورد استفاده برای بهینه سازی برمبنای معیار بهینگی کلاسیک استوار است که از استراتژی یک مجموعهٔ فعال یرای نهایت حد محدودیت ها روی متغیرهای طراحی استفاده میکند. ازیک روش تکراری با نقطه ثابت برمبنای این معیار که در نقطه بهینه انرژی کششی غیر خطی برای همهٔ اجزاء باید باهم مساوی باشند استفاده شده است. مثالهای متعددی ارائه شده است که درآن فرضیات انجام شده در روش قابل محاسبه و ارزیابی است و میتواند برخی از مشخصات روش پیشنهادی را بیان نماید.

INTRODUCTION

The minimum weight design of structures subject to a stability constraint is one of the most important problems in structural optimization and has attracted a great deal of interest in the structural mechanics community. Methods for optimum design of structures have progressed rapidly in recent years. In particular, optimality criterion procedures have significantly advanced the state-of-the-art of the minimum weight design of structures involving large finite-element assemblies.

Quite frequently, studies in the literature reveal that a linearized stability approach is used beyond its limits of applicability. ''Linear stability'' can only give physically significant answers if the linear analysis gives such deformations that are exactly the same when geometric nonlinearity is considered. This only happens in a very limited number of practical situations, such as a perfectly straight column under an axial load. In real engineering situations where the qualitative nature of the behavior is completely unknown, linearized stability represented by critical eigenmodes does not provide an adequate representation of nonlinear behavior of a structure. For such cases, it is more appropriate to optimize a structure on the basis of a ''nonlinear stability constraint'' when the structure has an inherent tendency.to possess nonlinear behavior.

Khot and Kamat [1] were among the first researchers who developed an optimization method based on an optimality criterion to minimize the design weight under system nonlinear stability for truss structures. Later, Kamat and Ruangsilasingha [2] and Kamat [3] formulated stability in its nonlinear form. They resorted to solving two special cases. They addressed the problem of maximization of the critical load of shallow space trusses and shallow truss arches of a given configuration and volume.

Levy and perng [4] discussed the optimal design of trusses to withstand nonlinear stability requirements. They developed a two-phase iterative procedure of analysis and design, where phase one utilized analysis to determine instability and phase two utilized a recurrence relation based on optimality criteria for redesign.

Virtually all of the methods developed in the area of linear and specially nonlinear stability optimal design procedures have dealt with the optimization of truss elements or truss like idealized elements, but are seldom engaged with complex stiffness elements such as three-dimensional beam elements. The present procedure develops an optimality criterion approach to determine the optimal minimum weight design of a structure idealized by three-dimensional beam elements with constraint on the nonlinear strain energy density distribution. This paper is an extension of the previous work, especially, the work done by Khot and Kamat [1].

NONLINEAR ANALYSIS

The approach employed for the analysis of three-dimensional structures for this study is based on a second-order approximation of the nonlinear equilibrium equations in terms of stress resultants. The kinematic hypothesis employed is based on the Bernoulli-Kirchhoff hypothesis that plane sections remain planar after deformation. On the basis of the assumption, the rigid body kinematic transformation can be expressed as

$$u_1(\mathbf{x}) = u - y\phi_3 + z\phi_2 \quad (1)$$

$$u_2(\mathbf{x}) = v - z\phi_1 \quad (2)$$

$$u_3(\mathbf{x}) = w + y\phi_1 \quad (3)$$

where the vector $\mathbf{u}_o = [u,v,w]^t$ is the displacement at the origin of axes (x,y,z) where (y=z=0) and $\phi_0 = [\phi_1, \phi_2, \phi_3]^t$ is the vector of rotation of cross section about (x,y,z) axes respectively. Note that superscript t means the transpose of the argument.

One can express Green's strain tensor in terms of Cartesian components of displacement vectors in indicial notation as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u'_{m,j}) \quad (4)$$

By using Equations 4, the following constitutive equations can be derived

$$N = EA\lambda_1 \quad \text{where} \quad \lambda_1 = u' + \frac{1}{2}[(u')^2 + (v')^2 + (w')^2] \quad (5)$$

$$N = GA_2\lambda_2 \quad \text{where} \quad \lambda_2 = v' - \phi_3(1+u') + w'\phi_1 \quad (6)$$

$$V_3 = GA_3\lambda_3 \quad \text{where} \quad \lambda_3 = w' + \phi_2(1+u') - v'\phi_1 \quad (7)$$

$$M_2 = EI_2\lambda_4 \quad \text{where} \quad \lambda_4 = \phi'_3(1+u') - w'\phi'_1 \quad (8)$$

$$M_3 = EI_3\lambda_5 \quad \text{where} \quad \lambda_5 = \phi'_2(1+u') - v'\phi'_1 \quad (9)$$

$$T = GJ\lambda_6 \quad \text{where} \quad \lambda_6 = \phi'_1 + \frac{I_3}{J}\phi'_2\phi_3 - \frac{I_2}{J}\phi_2\phi'_3 \quad (10)$$

where $\lambda \equiv [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6]^t$ is the vector of strain measures which is conjugate to the stress resultant **R**.

To set the stage for the following developments, the displacements are denoted by $\mathbf{u} \equiv [u, v, w, \phi_1, \phi_2, \phi_3]^t$, the stress resultants by $\mathbf{R} = [N, V_2, V_3, M_2, M_3, T]^t$, the vector of the applied forces by $q \equiv [p, q_2, q_3, m_2, m_3, t]^t$ with p being the applied axial force, q_2 and q_3 being the applied shear forces in directions y and z, respectively, m_2, m_3 being the applied moments in directions y and z, respectively, and t being the applied torque.

Equation 5 through 10 furnish the simplest constitutive model in terms of the generalized strains. Clearly, the model (5) through (10) derives the strain energy potential of the *i*th group, W_i, defined as

$$W_i=\frac{1}{2}\int_0^L [A_i(\mathbf{x}_i)E_i\lambda_1^2+(A_2)_i(\mathbf{x}_i)G_i\lambda_2^2 + (A_3)_i(\mathbf{x}_i)G_i\lambda_3^2+E_i\bar{I}_i+(\mathbf{x}_i)\lambda_4^2+E_i\underline{I}_i(\mathbf{x}_i)\lambda_5^2 + J_i(\mathbf{x}_i)\lambda_6^2]\, d\chi \qquad (11)$$

where $A_i(\mathbf{x})$, $(A_2)_i(\mathbf{x})$, $(A_3)_i(\mathbf{x})$, $\bar{I}_i(\mathbf{x})$, $\underline{I}_i(\mathbf{x})$, $J_i(\mathbf{x})$ are the area, major shear area, minor shear area, major moment of inertia, minor moment of inertia, and torsion constant of the cross section for group *i*, respectively, all of which depend on the design variables. The term G_i is the shear modulus of group *i*.

Finally, the total potential energy of a structure can be simply expressed as

$$\Pi=\sum_{i=1}^{N} W_i-\int_0^L \eta^t\mathbf{q}\, ds \qquad (12)$$

where *N* is the total number of groups.

FORMULATION AND DEVELOPMENT OF THE OPTIMIZATION PROBLEM

Nonlinear critical load, which may be either a limit or a bifurcation point, can be characterized as the load that results in a loss of positive definiteness of the tangent stiffness matrix. For the optimization problem considered here, the load distribution applied to the structure is specified and is assumed to be proportional. The geometry of the structure is given and the optimization problem seeks to procure the minimum weight design of a structure, such that for the given design load distribution an instability can be achieved. The instability can be a limit or a bifurcation point. The structure is idealized with nonlinear beam elements and, in each optimization cycle, the structure is analyzed using the geometrically nonlinear procedure formulated earlier. By using a displacement control procedure such as Ramm (1980) and Batoz and Dhatt [5], one can reach and pass a limit or a bifurcation point. A limit or a bifurcation point can be traced either by monitoring the positive definiteness of tangent stiffness matrix or using the current stiffness parameter developed by Bergan and Soreide [6]. They showed that the nonlinear behavior of multidimensional problems may be characterized by a single scalar quantity called the *current stiffness parameter*. The current stiffness parameter is implemented and employed in this study. As soon as the load approaches an instability load, the current stiffness parameter approaches zero and becomes zero right at the instability load, changing sign after passing the instability point.

For the present study, the members of the structures are arranged into *M* distinct groups. Each group is associated with a set of design variables that describe the geometry of the cross section of that group. For example an I-beam can be described by its depth *h*, flange width *b*, web thickness *t*, flange thickness t_f. Consequently, the I-beam has four design variables. A rectangular cross section has two design variables: the width *b* and the height *h* of the cross section. A square cross section can be identified by one design variable which can be either the width or the height of the cross section. The vector of design variables will be designated as $\mathbf{x} = \{\chi_1, \chi_2, \ldots, \chi_{dv}\}$, where dv is the number of design variables, computed as the sum over all the groups of the number of design variables per group. Therefore, the optimization algorithm can handle complex cross sections with several variables to describe the cross section. One important point that needs to be mentioned here is that this study is not concerned with local buckling but rather is concerned with global instability of a structure. Because the analytical model does not include local buckling modes, these will not be represented in the objective function or constrain functions. One could use a model that incorporates local bucking, but the interaction between local and global buckling for most structures is small. On the other hand, since local and global buckling are lightly coupled, local buckling constraints in the form of the width-to-thickness limitation would be relatively simple to describe and implement.

To simplify notation, the *specific weight* of the *m*th group is designated as the weight per unit of the cross-sectional area of the entire group

$$q_m = \sum_{i \in m} \rho_i L_i \qquad (13)$$

where the length of member i is L_i and its density is ρ_i. The sum is taken over vall members associated with group m.

With the aobe definitions, the optimization problem can be posed in the following way

$$\text{Minimize} \quad \mathbf{q}^t\mathbf{a}(\mathbf{x}) \qquad (14)$$

$$\text{Such that} \quad h(\mathbf{x}) \equiv \Pi(\mathbf{x}) - \overline{\Pi} \equiv 0 \qquad (15)$$

$$\text{and} \quad \underline{\mathbf{x}} \leq \mathbf{x} \leq \overline{\mathbf{x}} \qquad (16)$$

where $\Pi(\mathbf{x})$ is the total potential energy and $\overline{\Pi}$ is the total potential energy associated with the optimum design at the nonlinear critical load. $\mathbf{q}= \{q_1, q_2, \dots, q_M\}$ is the vector of specific weights, and $\mathbf{a}(\chi)= \{a_1(\mathbf{x}), a_2(\mathbf{x}), \dots, a_m(\mathbf{x})\}$ is the vector of cross-sectional areas of M groups. Each function $a_m(\mathbf{x})$ depends only on the design variables from group m. The inequality constraints given in Equation 16 indicate that each design variable has a minimum permissible size, $\underline{\chi}_i$, and a maximum permissible size, $\overline{\chi}_i$.

Optimality Criteria

The optimality criteria can be obtained from the first-order necessary conditions for a constraint optimum. The Lagrangian functional corresponding to the optimization problem given in Equation 14 through 16 can be written as

$$L(\mathbf{x},\xi) = \mathbf{q}^t\mathbf{a}(\mathbf{x}) - \xi\,[\Pi - \overline{\Pi}] \qquad (17)$$

where ξ is the Lagrange multiplier for the equality constraint. The element size-limit constraints are not included in the Lagrangian and hence do not have the corresponding Lagrange multipliers. The explicit size constraints can be handled more efficiently with an active-set strategy. Whenever a design variable violates a size constraint, it is assigned its limiting value, is removed from the active set, and no longer is considered as a design variable.

The first-order necessary conditions for an optimum design are obtained by differentiating the Lagrangian functional with respect to design variables $\mathbf{x}$ and by setting the corresponding equation to zero

$$\nabla L(\mathbf{x}, \xi) = \mathbf{q}^t\nabla\mathbf{a}(\mathbf{x}) - \xi\,[\nabla\Pi - D\Pi.\mathbf{u}] = 0 \qquad (18)$$

where $[\nabla(.)]_j = d(.)/\partial\chi_j$ is the ordinary gradient operator and $D\Pi.\mathbf{u}$ is zero. Therefore, Equation 18 yields

$$\mathbf{q}^t\nabla\mathbf{a}(\mathbf{x}) - \xi\,\nabla\Pi = 0. \qquad (19)$$

To simpplify the formulation, it is necessary to make some definitions. First define the vector $\mathbf{F}$ to be the gradient of the objective function

$$\mathbf{F}(\mathbf{x}) = \mathbf{q}^t\nabla\mathbf{a}(\mathbf{x}) = \mathbf{q}^t A(\mathbf{x}) \qquad (20)$$

where the components of the newly defined matrix $\mathbf{A}$ are given by $\mathbf{A} = \partial\alpha_i/\partial\chi_j$, Further, define the vector $\mathbf{P}$ to be the gradient of the constriant

$$\mathbf{P}(\mathbf{x}) = \nabla\Pi. \qquad (21)$$

Thus, the optimality criteria presented by Equation 18 simply reduces to

$$\mathbf{F}(\mathbf{x}) - \xi\, \mathbf{P}(\mathbf{x}) = 0. \tag{22}$$

This equation geometrically signifies that, to have an optimal solution **x**, for a design the vectors **F** and **P** must be collinear.

Later on, the derived optimality criteria, Equation 22, will be used to set up the recurrence algorithm for updating the design vector **x**. To simplify the notation for future developments, one can define a diagonal quotient matrix **Q** such that

$$Q_{ij} = \begin{cases} \dfrac{F_i}{\xi P_i} & \text{if } i = j \\ 0 & \text{if } i \neq j\,. \end{cases} \tag{23}$$

Therefore, by using the definition given in Equation 23 and by considering the optimality criteria given by Equation 22, one will obtain the following simplified optimality criteria expression

$$\mathbf{Q}(\mathbf{x}) = \mathbf{I} \tag{24}$$

where **I** is the identity matrix.

SOLUTION PROCEDURE

The optimum design must statisfy the optimality criteria and the nonlinear energy density constraint. Since these equations are nonlinear, they must be solved by an iterative procedure. The algorithm used here is a fixed-point iteration based on the first order necessary conditions (optimality criteria). The fixed-point iteration, used in conjunction with a scalling procedure, will move the initial design toward a configuration which satisfies the optimality criteria and the constraints. The algorithm steps are as follows:

1) Choose an initial design;
2) Perform a nonlinear analysis and determine the nonlinear limit load;
3) Perform a line search (scaling) to satisfy the constraint and keep the design in feasible region, by assuming the Lagrange multiplier equal to unity;
4) Select a design vector direction and determine new design variables;
5) Check the optimality criteria;
6) If convergence is not acheived, go to step 2.

The Fixed-Point Iteration

Various forms of recurrence relations have been developed by various researchers and have been used to update the configuration in an optimization problem (Gellaty and Berke [7], and Khot, Venkayya and Berke [8]). The recursive approach has the advantage of eliminating the need for the Hessian of the Lagrangian functional in a nonlinear programming algorithm. In general, the optimality criteria are used to modify the design variable vector. Therefore, one can generate a new design vector from the previous one with the exponential recursion relation

$$\mathbf{x}^{k+1} = \mathbf{x}^{k}\, [\mathbf{Q}(\mathbf{x}^{k})]^{\frac{1}{r}} \tag{25}$$

where k denotes the iteration number and r is the step size parameter. At the optimum, the optimality criteria will be satisfied and therefore, the design variables will be unchanged with any additional iterations. The convergence behavior depends on the parameter r. Depending on the behavior of the constraint, it may be necessary to increase r to prevent convergence.

At the optimum, the term $\mathbf{Q}(\mathbf{x}^k)$ given in Equation 25 approaches identity. Therefore, linearizing the exponential gives the alternate recurrence relation

$$x^{k+1} = x^{k}\left[I + \frac{1}{r}[Q(x^{k}) - I]\right] \quad (26)$$

This equation is referred to as the linear recurrence relation for the design variables and can be used to update the design.

Active-Set Constraint Strategy

After each iteration, a new set of design variables is obtained. If a design variable lies within its permissible range, it is placed in the active set; otherwise, it is placed in the passive set so that a proper scaling can be performed before the next iteration. At the start of each iteration, a formerly passive variable can either remain in the passive set or be reactivated. In general, it is not known apriori if a variable will be active at the optimum.

Sensitivity Analysis

Evaluation of the optimality conditions, Equation 22, requires knowledge of the sensitivity, or rate of change, of the potential energy with respect to the design variables. The sensitivity of the potential energy can be computed by differentiating the potential energy given in Equation 12 with respect to the design variable, **x**,

$$\nabla\Pi = \nabla W \quad (27)$$

in which the differentiation of the strain energy involves only differentiation of the scalar cross-sectional parameters of Equation 11.

$$\nabla W_i = \frac{1}{2}\int_0^L [\nabla A_i(\mathbf{x}_i) E_i \lambda_i^2 + \nabla(A_2)_i(\mathbf{x}_i) G_i \lambda_2^2 + \nabla(A_3)_i(\mathbf{x}_i) G_i \lambda_3^2 + \nabla \bar{I}_i E_i(\mathbf{x}_i)\lambda_4^2 + \nabla \underline{I}_i E_i(\mathbf{x}_i)\lambda_5^2 + \nabla J_i(\mathbf{x}_i)\lambda_6^2]\, d\chi - \int_0^L \nabla\eta^t \mathbf{q}\, d\chi \quad (28)$$

where $\bar{\mathbf{I}} \equiv [\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_m]^t$ is the vector of major moment of inertias of M groups, $\underline{\mathbf{I}} = [\underline{\mathbf{I}}_1, \underline{\mathbf{I}}_2, \ldots, \underline{\mathbf{I}}_m]^t$ is the vector of minor moment of inertias of M groups, $\mathbf{J} = [J_1, J_2, \ldots, J_m]^t$ is the vector of torsion constants of M groups. Each function such as $\underline{\mathbf{I}}_m(\mathbf{x})$ depends only on the design variables from group m. The sensitivities of λ_i are identically zero for statically determinate structures since the distribution of force through the structure does not depend on the element rigidities. In addition, such sensitivities are usually small and are generally neglected in practical computations for indeterminate structures.

The following section describes how the developed nonlinear analysis procedure and the proposed optimization method are applied to several simple structures. The purpose of the examples is to demonstrate how the algorithm performs.

Examples

In the example investigated in this report, Young's modulus of elasticity of 10^7 psi and material density, ρ, of the 0.1 lbs/in^3 were used. A square cross section was used for all the members with a minimum allowable size of 0.316 inches for each design variable (or 0.1 in^2 for minimum allowable cross-sectional area). Each design variable consists of either height or width of a square cross-section. In addition, each structural member is modeled with 2 three-noded quadratic geometrically nonlinear beam elements.

The analysis and the optimization procedure, used in the analysis and design of the structures studied here, are implemented in the general finite element program FEAP [10].

Four-Beam Structure

The four-beam structure consists of four beams with four fixed supports, and all the beams have one

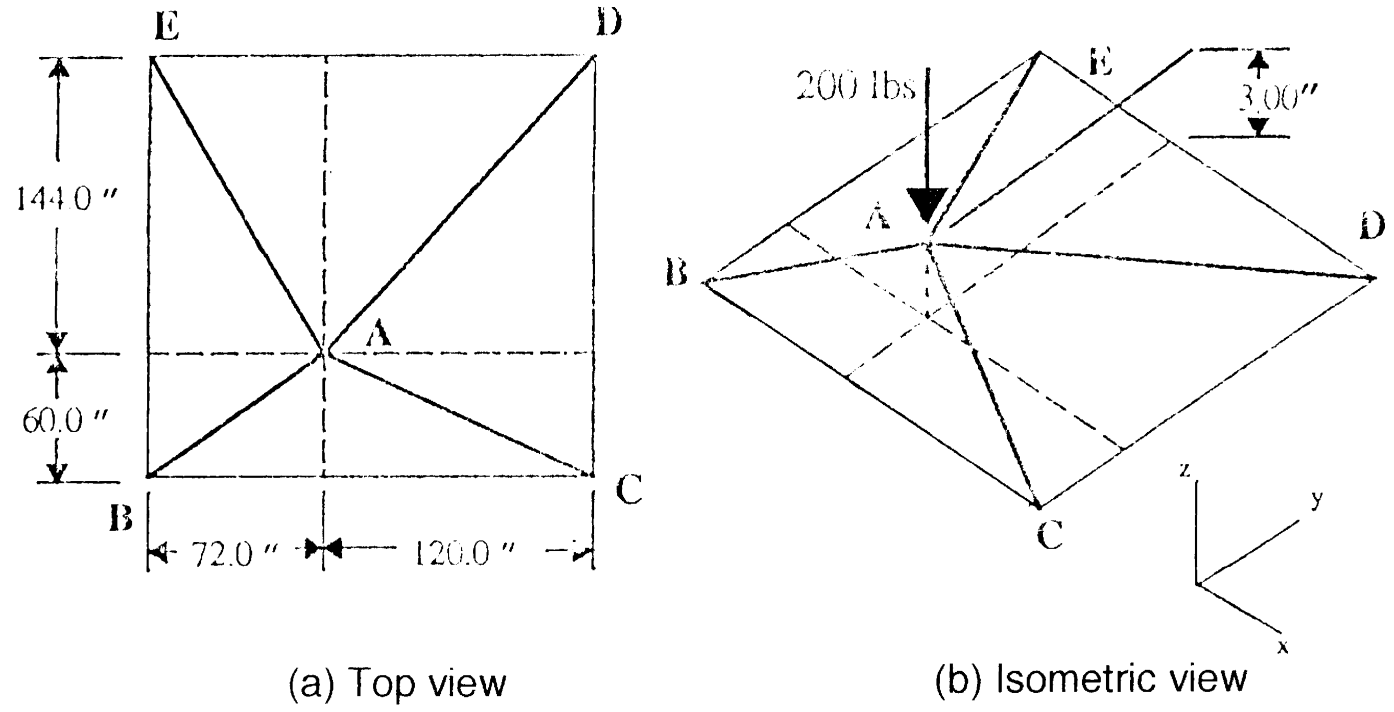

Figure 1. Topology of four-beam structure

common end point. The topology of the structure is given in Figure 1(a) and 1(b). The loading on the structure consists of a single proportional load applied at junction node in negative z-direction. The magnitude of the applied loading is 200 lbs. This struture is to be optimized to determine the minimum weight optimal design that exhibits a limit or a bifurcation instability under the applied loading. The initial design was chosen with the cross-sectional areas of each member to be 2.0929 in^2, with a total initial weight of 120.64 lbs. The history of the optimization process is given in Figure 2. The problem converged in 15 iterations with the cross-sectional properties as given in Table 1. The weight of the structure at the optimum is 77.67 lbs.

The relative strain energy densities of each member are also given in Table 1. One of the members became passive at the 11th iteration, and the other three members remained active with strain energy densities of almost unity. The member that became passive is the member AD (see Figures 1(a) and 1(b)). Member AD is the longest member of the structure and consequently, the most slender member of the system. The optimization problem, converged with member AD becoming passive, physically points out that this member is an unimportant member and that

TABLE 1. Properties of Four-Beam Structure

Member	Cross Sectional Area (in^2)	Major moment of Inertia (in^4)	Minor Moment of inertia (in^4)	Torsion (in^4)	Relative Energy Densities at Optimum
AB	2.7101	0.61207	0.61207	10.356	1.0000
AC	1.8762	0.29336	0.29336	4.963	0.99786
AD	0.1000	0.00083	0.00083	0.013	0.20986*
AE	1.5665	0.20423	0.20423	3.455	0.99770

*Passive element

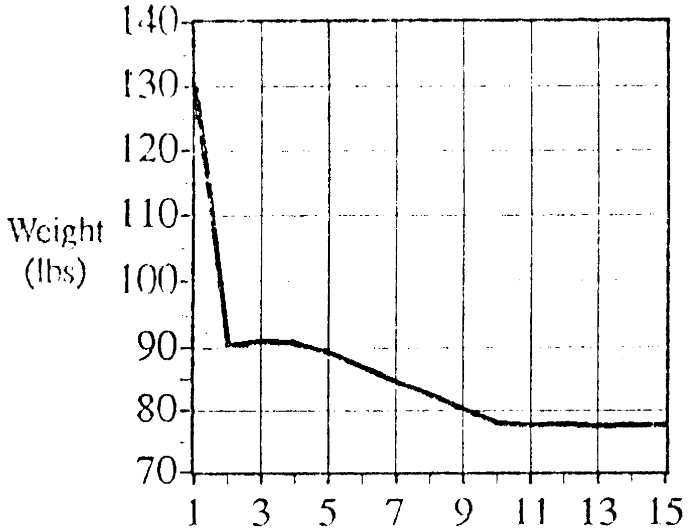

Figure 2. Optimization cycles versus weight

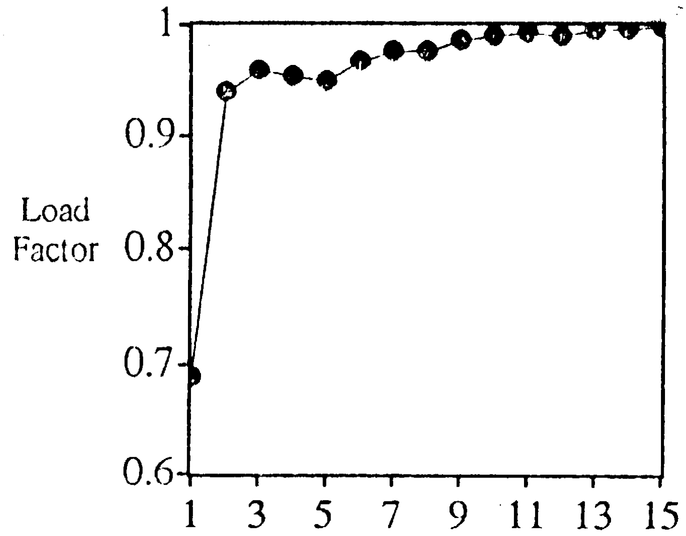

Figure 3. Optimization cycles versus load factor

in order to have a minimum weight structure to exhibit a limit or a bifurcation instability under the applied loading, one needs to take this elemen out of the active-set. This was done by the algorithm by assigning the minimum allowable design variable size to member AD and setting it to be a passive element.

Figure 3 presents the magnitude of the load factor at limit load versus the optimization cycle. Looking at this figure, it becomes apparent that the structure is reallocating properties to the different elements to achieve the minimum weight and unity for the load factor at the limit load. After the second iteration, the optimization problem moves smoothly toward having unity for the load factor at the limit load, which happens at the fifth iteration.

CONCLUSIONS

An optimization-based design that efficiently produces structural design with minimum weight and with geometric nonlinear behavior has been presented. The procedure is based on nonlinear stability for structures idealized with two-and three-dimensional beam elements. The method establishes a rational framework to address the nonlinear stability of the minimum weight design of structures whcih exhibit instability under the applied loading conditions.

REFERENCES

1. N. S. Khot and M. P. Kamat, "Minimum Weight Design of Structures with Geometric Nonlinear Behavior", *AIAA paper No. 83-0973, AIAA/ASME/ASCE/AHS 25th Structures, Structural Dynamics, and Material Conference*, Lake Tahoe, Nevada, May, (1983), 383-391.
2. M. P. Kamat and P. Ruangsilasingha, "Optimization of Space Trusses Against Instability Using Design Sensitivity Derivatives": *Engineering Optimization*, 8, (1985), 177-188.
3. M. P. Kamat, "Optimization of Shallow Arches Against Instability Using Design Sensitivity Derivatives": *Finite Element in Analysis and Design*, 3, (1987), 277-284.
4. R. Levy, and H. S. Perng, "Optimization for Nonlinear Stability": *Computers & Structures*, 30(3), (1988), 529-535.
5. J. L. Batoz and G. Dhatt, "Incremental Displacement Algorithms for Nonlinear Problems": *International Journal for Numerical Methods in Engineering*, 14, (1971), 1262-1267.
6. P. G. Bergan and T. H. Soreide, "Solution of Large Displacements and Instability Problems Using the Current Stiffness Parameter": *Finite Elements in Nonlinear*

Mechanics, 2, TAPIR, Norway, (1978).

7. R. A. Gellaty and L. Berke, "Optimal Structural Design" *Report No. AFFDL-TR-70-165*, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio, (1971).

8. N. S. Khot V. B. Venkayya and L. Berke, "Optimum Structural Design with Stability Constraints": *Int. J. Num. Meth. Engrg.*, 10, (1976), 1097-1114.

9. O. C. Zienkiewicz, "The Finite Element Method", 3rd Edition, McGraw-Hill, London, UK., (1982).

10. K. D. Hjelmstad and S. Pezeshk, "Approximate Analysis of Post-Limit Response of Frames": *ASCE J. Struct. Engtg.*, 114(2), (1988), 314-331.

11. K. D. Hjelmstad and S. Pezeshk, "Optimal Design of Frames to Resist Buckling Under Multiple Load Cases": *ASCE J. Struct. Engrg.*, 117(3), (1991), 914-935.

12. T. J. R. Hughes and K. S. Pister, "Consistent Linearization in Mechanics of Solids and Structures": *Computers and Structures*, 8(2), (1978), 391-397.

13. N. S. Khot "Algorithm Based on Optimality Criteria to Design Minimum Weight Structures": *Engineering Optimization*, 5, (1981), 73-90.

14. N. Olhoff and J. E. Taylor, "On Structural Optimization": *Journal of Applied Mechanics*, 50, (1983), 1139-1151.