NONRESONANT EXCITATION OF THE FORCED DUFFING EQUATION

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Abstract We investigate the hard nonresonant excitation of the forced Duffing equation with a positive damping parameter ε. Using the symbolic manipulation system MACSYMA, a computer algebra system, we derive the two term perturbation expansion by the method of multiple time scales. The resulting approximate solution is valid for small values of the coefficient ε. As the damping parameter ε increases, the accuracy of this solution degrades. In order to obtain an improved approximate solution to the given time dependent initial value problem, a hybrid perturbation-Galerkin method is applied to the perturbation solution. The hybrid method is based on Galerkin's method for determining an approximate solution to a differential equation using the perturbation solutions as trial functions. This hybrid method has the potential of overcoming some of the drawbacks of the perturbation method and the Galerkin method when they are applied separately, while combining some of the good features of both. We compare these two solutions for various values of ε and Ω (the frequency of the external force) and demonstrate the effectiveness of the hybrid method. Both the perturbation and hybrid solutions are also compared to a fourth order Runge-Kutta solution of the Duffing equation. For small values of ε, the hybrid solution is very close to the numerical solution for most values of Ω while the perturbation solution slightly overestimates the numerical solution. For larger values of ε, the perturbation solution deviates from the numerical solution very rapidly while the hybrid method remains close to the numerical model.

Key Words Nonresonant, Excitation, Duffing Equation, Damping Parameter, Perturbation

چکیده پدیده غیر همساز معادله دافینگ با نیروی محرک خارجی با پارامتر مثبت میرائی ε را مورد بررسی قرار می دهیم. با استفاده از سیستم محاسبه ای MACSYMA، که یک سیستم کامپیوتری جبری است، بسط دو جمله ای اختلالی را از روش چند گامی زمانی بدست می آوردیم. نتیجه تقریبی بدست آمده برای مقادیر کوچک ε قابل قبول است. چنانچه ضریب میرائی بزرگ می شود دقت جواب کم می گردد. برای بدست آوردن یک جواب تقریبی بهتر برای مسئله با مقدار اولیه وابسته به زمان یک روش اختلالی هایبرید-گالرکن را روی جواب اختلالی اعمال می کنیم. مبنای روش هایبرید، برای بدست آوردن یک جواب تقریبی معادله دیفرانسیل با بکار بردن جواب اختلالی بعنوان جواب اولیه (ازمایش)، روش گالرکن می باشد. روش هایبرید می تواند مقداری از کمبودهای روش اختلالی و روش گالرکن را، وقتی جداجدا اعمال شوند، با ترکیب مقداری از خصوصیات خوب هردو، جبران نماید. این دو جواب را برای مقادیر مختلف Ω (فرکانس نیروی خارجی) مقایسه می کنیم و موثر بودن روش هایبرید را نشان می دهیم. همچنین هر دو روش هایبرید و اختلالی با جواب بدست آمده از روش رونگ-کوتا برای بسیاری از مقادیر Ω به جواب عددی نزدیک تر است درحالیکه جواب اختلالی کمی بیشتر از جواب عددی می باشد. برای مقادیر بزرگتر Ω، جواب اختلالی به سرعت از جواب عددی منحرف می شود درحالیکه جواب هایبرید به جواب عددی نزدیک باقی می ماند.

INTRODUCTION

Perturbation solutions have been used successfully in a variety of differential equation type problems ([1], [3], [4], [5], [10], [11], and [13]). In constructing the perturbation solution, the usefulness of computer algebra systems have been realized and demonstrated by many investigators (see, e.g.s., [2] and [6]). The complexity of the perturbation solution increases as more terms are required. Although computer algebra systems alleviate some of the complications. It is still impractical to computer the higher order pertubation terms beyond a level. Unfortunately, in some applications , a large number of pertubation terms are required in order to obtain a reasonable approximation to the problem's solution. In such cases, one should try to make as much use as possible of information contained in the few lower order terms. The hybrid pertubration Galerkin method (which we will describe below) appears to increase the power and usefulness of the pertubration solution to a given problem [2], [7], [8].

The hybrid pertubation-Galerkin method is a two step technique, based on Galerkin's method for determining an approximate solution to a differential equation. In step one of the method, a formal perturbation solution of the problem $L(u, \varepsilon)=0$ is constructed, say of the form $u=\sum_{k=0}^{N-1} \varepsilon^k u_k + O(\varepsilon^N)$ which is formally valid as $\varepsilon \to 0$. In step two of the method, a improved approximate solution $\tilde{u}$ is sought in the form $\tilde{u}=\sum_{k=0}^{N-1} \lambda_k u_k$. The new "amplitudes" are obtained by requiring that $L(\tilde{u}, \varepsilon)$ is orthogonal to each of the coordinate function u_j, i.e.

$$\int L(\tilde{u}, \varepsilon)\, u_j(t)\, dt = 0, \quad j=0,1,\dots,N-1. \qquad (1)$$

Equation 1 is a system of N equations for the N unknown coefficients $\{\lambda_k\}$.

In this paper, we study the hard nonresonant excitation of the forced Duffing equations

$$\ddot{u} + 2\varepsilon\mu\dot{u} + \omega_0^2 u + \varepsilon\alpha U^3 = F \sin \Omega t,$$
$$u = 0 \ and\ \dot{u} = 0 \quad when \quad t = 0. \qquad (2)$$

In this equation $\dot{u}$ and $\ddot{u}$ are the first and second derivatives with respect to time, t, the parameters μ, ω_o, α, F, and Ω are treated as fixed constants, while $\varepsilon > 0$ is a positive damping parameter. We first develop a pertubation solution to the initial value problem (2) using the method of multiple time scales. The resulting solution is valid for small values of the perturbation parameter ε. As ε increases, the accuracy of the perturbation solution will degrade. In order to improve this solution, we will apply the hybrid perturbation-Galerk method just outlined.

HARD NORESONANT EXCITATION

For this case we assume $F=O(1)$ in Equation 2 and use the multiple time scales method [11] to construct a two term perturbation expansion of $u(t)$. We first define

$$T_0 = t \quad and \quad T_1 = \varepsilon t, \qquad (3)$$

and let

$$u(t) = u(T_0, T_1) = u_0(T_0, T_1) + \varepsilon u_1(T_0, T_1) + O(\varepsilon^2) \qquad (4)$$

Then

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} + O(\varepsilon^2),$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + o(\varepsilon^2) \qquad (5)$$

Using (3) - (5), Equation 2 becomes

$$(D_0^2 + 2\varepsilon D_0 D_1 + \dots)u + 2\varepsilon\mu(D_0 + \varepsilon D_1 + \dots)u + \omega_0^2 u + \varepsilon\alpha u^3$$
$$= F \sin \Omega t \qquad (6)$$

where $D_j = \frac{\partial}{\partial T_j}$. Substituting (4) for u gives

$$D_0^2 u_0 + \varepsilon D_0^2 u_1 + 2\varepsilon D_0 D_1 u_0 + 2\varepsilon\mu D_0 u_0 + \omega_0^2 u_0 + \varepsilon\omega_0^2 u_1 + \varepsilon\alpha u_0^3$$
$$+ O(\varepsilon^2) = F \sin \Omega T_0 \qquad (7)$$

Equating the coefficients of ε^0 and ε^1 respectively results in the following equations

$$D_0^2 u_0 + \omega_0^2 u_0 = F \sin \Omega T_0 \qquad (8)$$

$$D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 - 2\mu D_0 u_0 - \alpha u_0^3 \tag{9}$$

From (8) the solution u_0 is given by

$$u_0 = C(T_1)\cos(\omega_0 T_0 + \Phi(T_1)) + \frac{F}{\omega_0^2 - \Omega^2}\sin\Omega T_0 \tag{10}$$

where C and Φ are functions of T_1 and will be determined by requiring the elimination of the secular terms in higher order terms. Substituting (10) for u_0 into (9), we obtain the following differential equation for u_1.

$$D_0^2 u_1 + \omega_0^2 u_1 = 2\omega_0\left(\frac{dC}{dT_1} + \mu C\right)\sin(\omega_0 T_0 + \Phi)$$

$$+ 2C\left(\omega_o \frac{d\Phi}{dT_1} - 3\alpha\tilde{F}^2 - \frac{3}{8}\alpha C^2\right)\cos(\omega_o T_0 + \Phi)$$

$$- 3\alpha\tilde{F}(2\tilde{F}^2 + C^2)\sin\Omega T_0 - 4\mu\Omega\tilde{F}\cos\Omega T_0$$

$$+ 2\alpha\tilde{F}^3\sin 3\Omega T_0 - 3\alpha C^2\tilde{F}\sin\Omega T_0\cos(2(\omega_o T_0 + \phi))$$

$$+ 6\alpha C\tilde{F}^2\cos 2\Omega T_0\cos(\omega_o T_0 + \Phi)$$

$$- \frac{\alpha C^3}{4}\cos(3(\omega_0 T_0 + \Phi)) \tag{11}$$

where

$$\tilde{F} = \frac{1}{2}\frac{F}{\omega_0^2 - \Omega^2}$$

To eliminate the secular terms, we set the coefficients of $\sin(\omega_0 T_0 + \Phi)$ and $\cos(\omega_0 T_0 + \Phi)$ equal to zero, i. e.

$$2\omega_o\left(\frac{dC}{dT_1} + \mu C\right) = 0 \tag{12}$$

and

$$2C\left(\omega_o \frac{d\Phi}{dT_1} - 3\alpha\tilde{F}^2 - \frac{3}{8}\alpha C^2\right) = 0 \tag{13}$$

From (12) we find

$$C = \tilde{C}e^{-\mu T_1} \tag{14}$$

where $\tilde{C}$ may be treated as a constant.

Substituting (14) into (13) and solving the differential equation for Φ, we have

$$\Phi = -\frac{3\alpha}{16\mu\omega_o}\tilde{C}^2 e^{-2\mu T_1} + \frac{3\alpha}{\omega_o}\tilde{F}^2 T_1 + \tilde{\Phi} \tag{15}$$

Substituting for C and Φ and using $T_0 = t$ and $T_1 = \varepsilon t$, the one term approximation u_0 from (10) can be expressed as

$$u_0 = \tilde{C}e^{-\varepsilon\mu t}\cos\left(\omega_o t - \frac{3\alpha}{16\mu\omega_o}\tilde{C}^2 e^{-2\varepsilon\mu t} + \frac{3\varepsilon\alpha}{\omega_o}\tilde{F}^2 t + \tilde{\Phi}\right)$$

$$+ 2\tilde{F}\sin\Omega t \tag{16}$$

and the first derivative $\dot{u}_0$ is

$$\dot{u}_0 = -\tilde{C}e^{-\varepsilon\mu t}\left(\omega_o + \frac{3\varepsilon\alpha}{8\omega_o}\tilde{C}^2 e^{-2\varepsilon\mu t} + \frac{3\varepsilon\alpha}{\omega_o}\tilde{F}^2\right)\sin(\omega_o t$$

$$- \frac{3\alpha}{16\mu\omega_o}\tilde{C}^2 e^{-2\varepsilon\mu t} + \frac{3\varepsilon\alpha}{\omega_o}\tilde{F}^2 t + \tilde{\Phi}) - \varepsilon\mu\tilde{C}e^{-\varepsilon\mu t}\cos(\omega_o t$$

$$- \frac{3\alpha}{16\mu\omega_o}\tilde{C}^2 e^{-2\varepsilon\mu t} + \frac{3\varepsilon\alpha}{\omega_o}\tilde{F}^2 t + \tilde{\Phi}) + 2\tilde{F}\Omega\cos\Omega t \tag{17}$$

Applying the initial condition $u_0(0) = 0$ to (16) will result in

$$\tilde{\Phi} = \frac{3\alpha\tilde{C}^2}{16\mu\omega_o} - \frac{\pi}{2} \tag{18}$$

This relation expresses $\tilde{\Phi}$ in terms of $\tilde{C}$. Similarly, we apply the initial condition $\dot{u}_0(0) = 0$ to (17) and obtain

$$\frac{3\varepsilon\alpha}{\omega_0}\tilde{C}\tilde{F}^2 + \frac{3\varepsilon\alpha}{8\omega_o}\tilde{C}^3 + \omega_o\tilde{C} + 2\tilde{F}\Omega = 0 \tag{19}$$

which is a cubic equation and has three solutions for $\tilde{C}$.

The one term approximation for u can now be written as

$$u = \tilde{C}e^{-\varepsilon\mu t}\sin\left[\omega_o t + \frac{3\alpha}{16\mu\omega_o}\tilde{C}^2(1 - e^{-2\varepsilon\mu t})\right.$$

$$\left. + \frac{3\varepsilon a}{4\omega_0}\frac{F^2 t}{(\omega_0^2 - \Omega^2)^2}\right] + \frac{F}{\omega_0^2 - \Omega^2}\sin\Omega t + O(\varepsilon) \tag{20}$$

with $\tilde{C}$ the solution of (19).

After elimination of secular terms, the differential equation (11) for u_1 becomes

$$D_0^2 u_1 + \omega_0^2 u_1 = -3\alpha\tilde{F}(2\tilde{F}^2 + C^2)\sin\Omega T_0 - 4\mu\Omega\tilde{F}\cos\Omega T_0$$

$$+ 2\alpha\tilde{F}^3\sin 3\Omega T_0 - 3\alpha C^2\tilde{F}\sin\Omega T_0\cos(2(\omega_o T_0 + \Phi))$$

$$+ 6\alpha C\tilde{F}^2\cos 2\Omega T_0\cos(\omega_o T_0 + \Phi) - \frac{\alpha C^3}{4}\cos(3(\omega_o T_0 + \Phi))$$

$$\tag{21}$$

Using the symbolic manipulation system MACSYMA, the particular and homogeneous solutions for u_1 are given by

$$u_{1p} = -\frac{1}{32}\frac{\alpha C^3}{\omega_0^2}\cos(3(\omega_0 T_0+\Phi)) + \frac{3}{2}\alpha C^2 \frac{\tilde{F}}{(\omega_0+\Omega)(3\omega_0+\Omega)}$$

$$\sin(2(\omega_o T_0+\Phi)+\Omega T_0) - \frac{3}{2}\alpha C^2 \frac{\tilde{F}}{(\omega_o-\Omega)(3\omega_o-\Omega)}$$

$$\sin(2(\omega_o T_0+\Phi)-\Omega T_0) - \frac{3}{4}\alpha C \frac{\tilde{F}^2}{\Omega(\omega_o+\Omega)}$$

$$\cos(\omega_o T_0+\Phi+2\Omega T_0) + \frac{3}{4}\alpha C \frac{\tilde{F}^2}{\Omega(\omega_o-\Omega)}$$

$$\cos(\omega_o T_0+\Phi-2\Omega T_0) + 2\alpha\frac{\tilde{F}^3}{(\omega_o^2-9\Omega^2)}\sin 3\Omega T_0$$

$$-3\alpha(2\tilde{F}^2+C^2)\frac{\tilde{F}}{(\omega_o^2-\Omega^2)}\sin\Omega T_0 - 4\mu\Omega\frac{\tilde{F}}{(\omega_o^2+\Omega^2)}\cos\Omega T_0 \qquad (22)$$

and

$$u_{1h} = C_1\sin\omega_o T_0 + C_2\cos\omega_o T_0 \qquad (23)$$

where C_1 and C_2 are constants and will be determined by applying the initial conditions. The total solution for u_1 is

$$u_1 = u_{1p} + u_{1h}. \qquad (24)$$

Substituting for C, Φ, and $\tilde{\Phi}$ from (14), (15), and (18) and then using $T_0 = t$ and $T_1 = \varepsilon t$, u_1 can be expressed as

$$u_1 = -\frac{1}{32}\frac{\alpha\tilde{C}^3}{\omega_o^2}e^{-3\varepsilon\mu t}\sin[3(\omega_o t + \frac{3\alpha\tilde{C}^2}{16\mu\omega_o}(1-e^{-2\varepsilon\mu t}) + \frac{3\varepsilon\alpha}{\omega_o}\tilde{F}^2 t)]$$

$$-\frac{3}{2}\frac{\alpha\tilde{C}^2 e^{-2\varepsilon\mu t}\tilde{F}}{(\omega_o+\Omega)(3\omega_o+\Omega)}\sin[2(\omega_o t + \frac{3\alpha\tilde{C}^2}{16\mu\omega_o}(1-e^{-2\varepsilon\mu t})$$

$$+\frac{3\varepsilon\alpha}{\omega_o}\tilde{F}^2 t) + \Omega t] + \frac{3}{2}\frac{\alpha\tilde{C}^2 e^{-2\varepsilon\mu t}\tilde{F}}{(\omega_o-\Omega)(3\omega_o-\Omega)}\sin[2(\omega_o t$$

$$+\frac{3\alpha\tilde{C}^2}{16\mu\omega_o}(1-e^{-2\varepsilon\mu t}) + \frac{3\varepsilon\alpha}{\omega_o}\tilde{F}^2 t) - \Omega t] - \frac{3}{4}\frac{\alpha\tilde{C}e^{-\varepsilon\mu t}\tilde{F}^2}{\Omega(\omega_o+\Omega)}$$

$$\sin[\omega_0 t + \frac{3\alpha\tilde{C}^2}{16\mu\omega_0}(1-e^{-2\varepsilon\mu t}) + \frac{3\varepsilon\alpha}{\omega_0}\tilde{F}^2 t) + 2\Omega t]$$

$$+\frac{3}{4}\frac{\alpha\tilde{C}^2 e^{-\varepsilon\mu t}\tilde{F}^2}{\Omega(\omega_o-\Omega)}\sin[\omega_o t + \frac{3\alpha\tilde{C}^2}{16\mu\omega_o}(1-e^{-2\varepsilon\mu t}) + \frac{3\varepsilon\alpha}{\omega_o}\tilde{F}^2 t$$

$$-2\Omega t] + \frac{2\alpha\tilde{F}^3}{(\omega_o^2-9\Omega^2)}\sin 3\Omega t - 3\alpha(2\tilde{F}^2+\tilde{C}^2 e^{-2\varepsilon\mu t})\frac{\tilde{F}}{(\omega_o^2-\Omega^2)}$$

$$\sin\Omega t - \frac{4\mu\Omega\tilde{F}}{(\omega_o^2-\Omega^2)}\cos\Omega t + C_1\sin\omega_o t + C_2\cos\omega_o t \qquad (25)$$

Differentiating u_1 with respect to t, an expression for $\dot{u}_1$ is obtained. Then, we apply the initial conditions $u_1(0) = 0$ and $\dot{u}_1(0) = 0$ to the resulting expressions for u_1 and $\dot{u}_1$ and solve for the undetermined constants C_1 and C_2. From (25) we obtain

$$C_2 = 4\mu\Omega\frac{\tilde{F}}{(\omega_o^2-\Omega^2)} \qquad (26)$$

and from $\dot{u}_1(0) = 0$ we have

$$C_1 = \frac{1}{\omega_o}\dot{u}_{1p}(0) = \frac{3}{32}\frac{\alpha\tilde{C}^3}{\omega_o^3}(\omega_o + \frac{3\varepsilon\alpha\tilde{F}^2}{\omega_o} + \frac{3\varepsilon\alpha\tilde{C}^2}{8\omega_o})$$

$$+\frac{3}{2}\frac{\alpha\tilde{C}^2\tilde{F}}{\omega_o(\omega_o+\Omega)(3\omega_o+\Omega)}(2\omega_o + \frac{6\varepsilon\alpha\tilde{F}^2}{\omega_o} + \frac{3\varepsilon\alpha\tilde{C}^2}{4\omega_o} + \Omega)$$

$$-\frac{3}{2}\frac{\alpha\tilde{C}^2\tilde{F}}{\omega_o(\omega_o-\Omega)(3\omega_o-\Omega)}(2\omega_o + \frac{6\varepsilon\alpha\tilde{F}^2}{\omega_o} + \frac{3\varepsilon\alpha\tilde{C}^2}{4\omega_o} - \Omega)$$

$$+\frac{3}{4}\frac{\alpha\tilde{C}\tilde{F}^2}{\omega_o\Omega(\omega_o+\Omega)}(\omega_o + \frac{3\varepsilon\alpha\tilde{F}^2}{\omega_o} + \frac{3\varepsilon\alpha\tilde{C}^2}{8\omega_o} + 2\Omega)$$

$$-\frac{3}{4}\frac{\alpha\tilde{C}\tilde{F}^2}{\omega_o\Omega(\omega_o-\Omega)}(\omega_o + \frac{3\varepsilon\alpha\tilde{F}^2}{\omega_o} + \frac{3\varepsilon\alpha\tilde{C}^2}{8\omega_o} - 2\Omega)$$

$$-\frac{6\alpha\Omega\tilde{F}^3}{\omega_o(\omega_o^2-9\Omega^2)} + 3\alpha\Omega(\tilde{C}^2+2\tilde{F}^2)\frac{\tilde{F}}{\omega_o(\omega_o^2-\Omega^2)} \qquad (27)$$

Substituting the expressions for C_1 and C_2 into (25) will result in the total solution for u_1.

The two term approximation for u is

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2) = \tilde{C}e^{-\varepsilon\mu t}\sin[\omega_o t + \frac{3\alpha}{16\mu\omega_o}\tilde{C}^2(1-e^{-2\varepsilon\mu t})$$

$$+\frac{3\varepsilon\alpha}{4\omega_o}\frac{\tilde{F}^2 t}{(\omega_o^2-\Omega^2)^2}] + \frac{F}{\omega_o^2-\Omega^2}\sin\Omega t$$

$$+\varepsilon(C_1\sin\omega_o t + C_2\cos\omega_o t + u_{1p}) + O(\varepsilon^2) \qquad (28)$$

where u_{1p} is given in (25).

THE HYBRID PERTUBRATION-GALERKIN METHOD

In general terms, given the differential equation

$$L(u,\varepsilon) = 0 \qquad (29)$$

where ε is a small parameter, the perturbation solution to (29) can be expressed (in the case of a regular expansion) as

$$u=\sum_{k=0}^{N-1}\varepsilon^k u_k+O(\varepsilon^N) \qquad (30)$$

The hybrid Galerkin method is a two-step analysis technique [6, 7]. The first step involves the computation of the perturbation solution in form of (30) for a particular problem of type (29). The perturbation functions u_k in (30) are determined from a series of equations obtained by substituting (30) into (29) and setting the coefficients of ε^k equal to zero, for $k=0, 1, ..., N-1$. In the second step, new amplitudes of the perturbation coordinate functions u_k are computed by using Galerkin method. Thus, an improved approximate solution $\tilde{u}$ for u is sought in the form

$$\tilde{u}=\sum_{k=0}^{N-1}\lambda_k u_k+O(\varepsilon^N) \qquad (31)$$

where the N unknown parameters $\lambda_k=\lambda_k(\varepsilon)$ represent the amplitudes of the coordinate functions u_k. To determine these parameters, we substitute $\tilde{u}$ into the given differential equation (29) and require that the residual is orthogonal to each perturbation coordinate function, i. e.

$$\int_0^\tau L(\tilde{u},\varepsilon)\,u_j(t)\,dt=0 \quad for \quad j=0,1,...,N-1. \qquad (32)$$

Equation 32 represents a set of N simultaneous equations for the N unknown amplitudes $\lambda_0, \lambda_1, ..., \lambda_{N-1}$.

The differential equation (29) for the forced Duffing equation (2) can be written as

$$L(u,\varepsilon)=\ddot{u}+2\varepsilon\mu\dot{u}+\omega_0^2 u+\varepsilon\alpha u^3-F\sin\Omega t=0. \qquad (33)$$

For a one term perturbation solution u_0, a hybrid solution $\tilde{u}=\lambda_0 u_0$ is obtained from (31). Substituting $\tilde{u}$ into (32) will result in the following equation:

$$\int_0^\tau [\lambda_0\ddot{u}_0+2\varepsilon\mu\lambda_0\dot{u}_0+\omega_0^2\lambda_0 u_0+\varepsilon\alpha\lambda_0^3 u_0^3-F\sin\Omega t]\,u_0\,dt=0 \qquad (34)$$

This equation can be rewritten as the following cubic equation for λ_0

$$\lambda_0^3\int_0^\tau \varepsilon\alpha u_0^4 dt+\lambda_0\int_0^\tau [u_0\ddot{u}_0+2\varepsilon\mu u_0\dot{u}_0+\omega_0^2 u_0^2]\,dt$$
$$-\int_0^\tau F u_0\sin\Omega t\,dt=0 \qquad (35)$$

Using an integration by parts, we have

$$\int_0^\tau u_0\ddot{u}_0 dt=u_0\dot{u}_0]_0^\tau-\int_0^\tau (\dot{u}_0)^2 dt \qquad (36)$$

Substitution of (36) into (35) eliminates the need to compute $\ddot{u}_0$ and hence (35) becomes

$$\lambda_0^3\int_0^\tau \varepsilon\alpha u_0^4 dt+\lambda_0\,\{u_0\dot{u}_0]_0^\tau+\int_0^\tau [-(\dot{u}_0)^2+2\varepsilon\mu u_0\dot{u}_0$$
$$+\omega_0^2 u_0^2]\,dt\,\}-\int_0^\tau F u_0\sin\Omega t\,dt=0 \qquad (37)$$

For a two term perturbation solution $u=u_0+\varepsilon u_1$, a hybrid solution $\tilde{u}=\lambda_0 u_0+\lambda_1 u_1$ obtained from (31). Substituting $\tilde{u}$ into (32) will result in the following equations for λ_0 and λ_1.

$$\lambda_0\int_0^\tau [u_0\ddot{u}_0+2\varepsilon\mu u_0\dot{u}_0+\omega_0^2 u_0^2]dt+\lambda_0^3\int_0^\tau \varepsilon\alpha u_0^4 dt$$
$$+\lambda_1\int_0^\tau [u_0\ddot{u}_1+2\varepsilon\mu u_0\dot{u}_1+\omega_0^2 u_0 u_1]dt+\lambda_1^3\int_0^\tau \varepsilon\alpha u_0 u_1^3 dt$$
$$+3\lambda_0^2\lambda_1\int_0^\tau \varepsilon\alpha u_0^3 u_1 dt+3\lambda_0\lambda_1^2\int_0^\tau \varepsilon\alpha u_0^2 u_1^2 dt$$
$$-\int_0^\tau F u_0\sin\Omega t\,dt=0 \qquad (38)$$

$$\lambda_0\int_0^\tau [u_1\ddot{u}_0 + 2\varepsilon\mu u_1\dot{u}_0 + \omega_0^2 u_0 u_1]\,dt + \lambda_0^3\int_0^\tau \varepsilon\alpha u_0^3 u_1\,dt$$

$$+\lambda_1\int_0^\tau [u_1\ddot{u}_1 + 2\varepsilon\mu u_1\dot{u}_1 + \omega_0^2 u_1^2]\,dt + \lambda_1^3\int_0^\tau \varepsilon\alpha u_1^4\,dt$$

$$+3\lambda_0^2\lambda_1\int_0^\tau \varepsilon\alpha u_0^2 u_1^2\,dt + 3\lambda_0\lambda_1^2\int_0^\tau \varepsilon\alpha u_0 u_1^3\,dt$$

$$-\int_0^\tau F u_1 \sin\Omega t\,dt = 0 \tag{39}$$

The $\ddot{u}_j$ terms can be eliminated by the identity

$$\int_0^\tau u_i\ddot{u}_j\,dt = u_i\dot{u}_j]_0^\tau - \int_0^\tau \dot{u}_i\dot{u}_j\,dt \tag{40}$$

and hence Equations 38 and 39 can be expressed as:

$$G_1 = a_1\lambda_0 + a_2\lambda_1 + a_3\lambda_0^3 + a_4\lambda_1^3 + a_5\lambda_0^2\lambda_1 + a_6\lambda_0\lambda_1^2 + a_7 = 0, \tag{41}$$

$$G_2 = b_1\lambda_0 + b_2\lambda_1 + b_3\lambda_0^3 + b_4\lambda_1^3 + b_5\lambda_0^2\lambda_1 + b_6\lambda_0\lambda_1^2 + b_7 = 0, \tag{42}$$

where $a_1, a_2, \ldots, a_7$ are given by

$$a_1 = u_0\dot{u}_0]_0^\tau + \int_0^\tau [-(\dot{u}_0)^2 + 2\varepsilon\mu u_0\dot{u}_0 + \omega_0^2 u_0^2]\,dt,$$

$$a_2 = u_0\dot{u}_1]_0^\tau + \int_0^\tau [-(\dot{u}_0\dot{u}_1) + 2\varepsilon\mu u_0\dot{u}_1 + \omega_0^2 u_0 u_1]\,dt,$$

$$a_3 = \int_0^\tau \varepsilon\alpha u_0^4\,dt, \quad a_4 = \int_0^\tau \varepsilon\alpha u_0 u_1^3\,dt,$$

$$a_5 = 3\int_0^\tau \varepsilon\alpha u_0^3 u_1\,dt, \quad a_6 = 3\int_0^\tau \varepsilon\alpha u_0^2 u_1^2\,dt,$$

$$a_7 = -\int_0^\tau F u_0 \sin\Omega t\,dt, \tag{43}$$

and $b_1, b_2, \ldots, b_7$ are given by

$$b_1 = u_1\dot{u}_0]_0^\tau + \int_0^\tau [-(\dot{u}_0\dot{u}_1) + 2\varepsilon\mu u_1\dot{u}_0 + \omega_0^2 u_0 u_1]\,dt,$$

$$b_2 = u_1\dot{u}_1]_0^\tau + \int_0^\tau [-(\dot{u}_1)^2 + 2\varepsilon\mu u_1\dot{u}_1 + \omega_0^2 u_1^2]\,dt,$$

$$b_3 = \int_0^\tau \varepsilon\alpha u_0^3 u_1\,dt, \quad b_4 = \int_0^\tau \varepsilon\alpha u_1^4\,dt,$$

$$b_5 = 3\int_0^\tau \varepsilon\alpha u_0^2 u_1^2\,dt, \quad b_6 = 3\int_0^\tau \varepsilon\alpha u_0 u_1^3\,dt,$$

$$b_7 = -\int_0^\tau F u_1 \sin\Omega t\,dt. \tag{44}$$

The two nonlinear Equations 41 and 42 must be solved numerically for the unknowns λ_0 and λ_1. To accomplish this, we use Newton's method. This iterative procedure is carried according to the relation

$$\vec{\lambda}^{(i+1)} = \vec{\lambda}^{(i)} - J(\vec{\lambda})^{-1}\,\vec{G}(\vec{\lambda}^{(i)}) \tag{45}$$

where $\vec{\lambda}^{(i)} = \begin{pmatrix}\lambda_0\\ \lambda_1\end{pmatrix}^{(i)}$ is the approximation at iteration i, $\vec{G}(\vec{\lambda}) = \begin{pmatrix}G_1\\ G_2\end{pmatrix}$ is a vector of nonlinear functions, and $J(\vec{\lambda})$ is Jacobian matrix of G defined as

$$J(\vec{\lambda}) = \begin{pmatrix} \dfrac{\partial G_1}{\partial\lambda_0} & \dfrac{\partial G_1}{\partial\lambda_1} \\ \dfrac{\partial G_2}{\partial\lambda_0} & \dfrac{\partial G_2}{\partial\lambda_1} \end{pmatrix} \tag{46}$$

with

$$\frac{\partial G_1}{\partial\lambda_0} = a_1 + 3a_3\lambda_0^2 + 2a_5\lambda_0\lambda_1 + a_6\lambda_1^2,$$

$$\frac{\partial G_1}{\partial\lambda_1} = a_2 + 3a_4\lambda_1^2 + a_5\lambda_0^2 + 2a_6\lambda_0\lambda_1,$$

$$\frac{\partial G_2}{\partial\lambda_0} = b_1 + 3b_3\lambda_0^2 + 2b_5\lambda_0\lambda_1 + b_6\lambda_1^2,$$

$$\frac{\partial G_2}{\partial\lambda_1} = b_2 + 3b_4\lambda_1^2 + b_5\lambda_0^2 + 2b_6\lambda_0\lambda_1.$$

For the first iteration we use $\vec{\lambda}^{(0)} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$. Then Equation 45 is repeated until $\left| \vec{\lambda}^{(i+1)} - \vec{\lambda}^{(i)} \right| \le \delta$ where δ is a predetermined small number, e.g. 10^{-6}.

RESULTS FOR THE HARD NONRESONANT PROBLEM

The hybrid perturbation - Galerkin method has been applied to the one and two term perturbation solutions as described previously. In this section we discuss the two term solution. In order to study the effects of ε and Ω on the solution to the forced Duffing equation (2), we have set the following parameters to predetermined constants as

$$F = 1,\ \alpha = 1,\ \omega_0 = 1,\ \mu = 0.2, \quad \tau = 10\pi \tag{47}$$

while varying the parameters ε and Ω. We have also applied the fourth-order Runge-Kutta method with a step size of $\pi/50$ to (2) and compared the perturbation and hybrid solutions to this numerical solution.

To obtain the nonresonant excitation only, the value of Ω must be away from $\omega_0 = 1.0$ (primary resonance). We have computed the perturbation, hybrid perturbation - Galerkin, and numerical solutions to the differential equation (2) for many different values of Ω and ε. In Figure 1 through Figure 4 the two - term perturbation and the two -

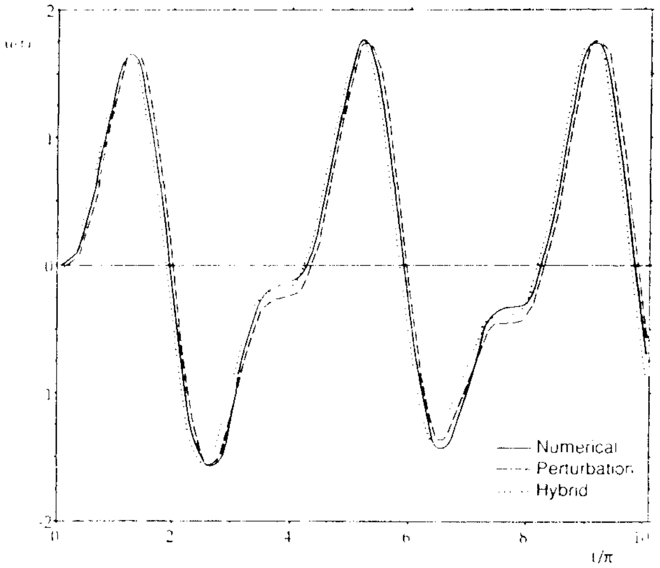

Figure 1. $\Omega = 0.5$, $\varepsilon = 0.02$

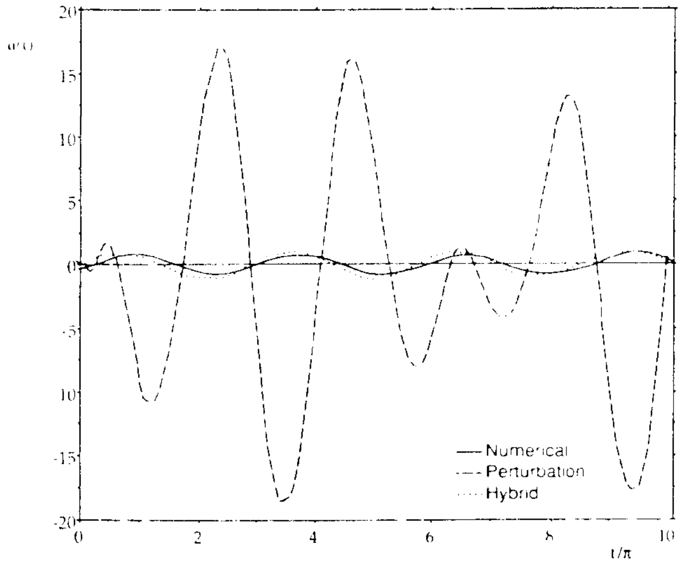

Figure 3. $\Omega = 0.7$, $\varepsilon = 1$

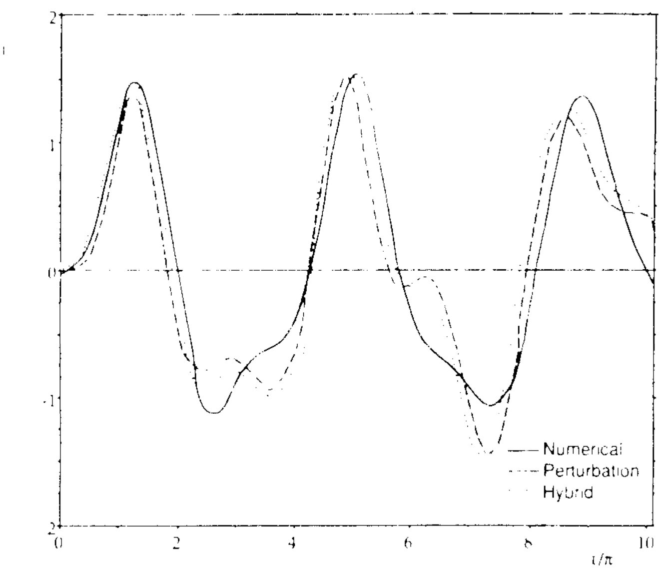

Figure 2. $\Omega = 0.5$, $\varepsilon = 0.1$

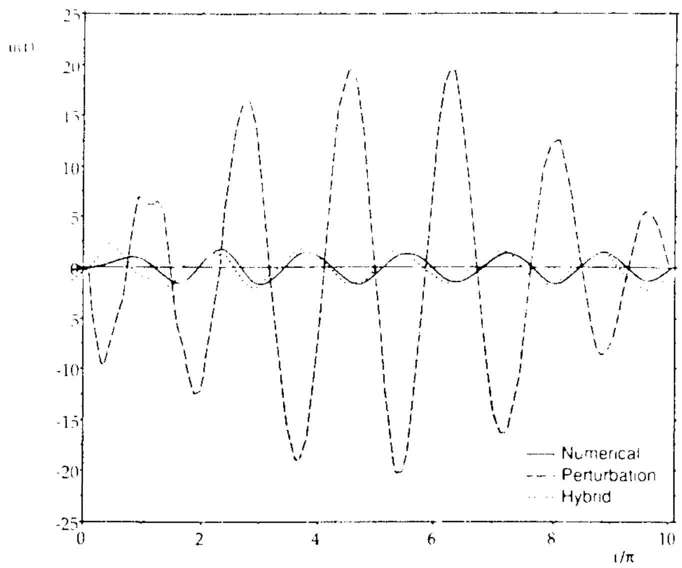

Figure 4. $\Omega = 1.2$, $\varepsilon = 0.5$

term hybrid solutions are compared with the numerical solution for some selected cases using the values in [47]. The first two figures illustrate the solutions for $\Omega = 0.5$ and $\varepsilon = 0.02, 0.10$. For small values of ε both the perturbation and hybrid solutions are very close to the numerical solution. As ε increases, we observe that the perturbation solution diverges from the numerical solution and the hybrid method results in a better approximation to the solution of (2).

As Ω approaches $\omega_0 = 1.0$ $(0.5 < \Omega < 1.0)$, we observe that the amplitude and shape of the perturbation solution deviates very rapidly from the numerical solution for increasing ε. For these cases, the hybrid solution improves the perturbation solution significantly. This is illustrated in Figure 3. When Ω increases above the primary resonance, the hybrid solution again provides a much better approximation to the solution of the differential equation (2) than the perturbation solution. Figure 4 is a representative of these cases. For values of $\Omega > 2.0$ while ε remains small (up to $\varepsilon = 0.5$) the perturbation and hybrid solutions are almost the same as the numerical solution. As ε increases, the hybrid method does not apear to improve the perturbation solution.

Table 1 contains the hybrid coefficients λ_j and λ_i for the hard nonresonant case for some selected values of Ω and ε.

TABLE 1

Ω	ε	λ_0	λ_1
0.5	0.02	0.9428	-0.0047
0.5	0.05	1.0324	0.0492
0.5	0.10	0.9974	0.0622
0.5	0.20	0.8690	0.0323
0.5	0.30	0.7992	0.0238
0.5	0.40	0.7965	0.0444
0.5	0.50	0.7230	0.0140
0.5	1.00	0.5951	-0.0074
0.7	0.20	0.9944	0.0520
0.7	0.50	0.5771	0.0037
0.7	0.75	0.5105	0.0017
0.7	1.00	0.4596	0.0007
0.8	0.10	1.3017	0.0383
0.8	0.20	0.9279	0.0227
0.8	0.50	0.4815	0.0030
0.8	1.00	0.3647	0.0009
0.9	0.05	1.3925	0.0074
0.9	0.10	0.9094	0.0044
0.9	0.50	0.2532	-0.0002
0.9	1.00	0.1762	-0.0002
1.2	0.02	-1.0939	-0.0328
1.2	0.10	-1.3011	0.0146
1.2	0.20	-1.3039	0.0340
1.2	0.50	-0.8566	0.0120
1.2	0.75	-0.6998	0.0066
1.2	1.00	-0.6095	0.0040
2.0	0.02	1.0009	0.0195
2.0	0.05	1.0016	0.0495
2.0	0.10	1.0029	0.1018
2.0	0.20	1.0048	0.2144
2.0	0.50	0.9971	0.5781
2.0	0.75	0.9748	0.8839
2.0	1.00	0.9402	1.1798

CONCLUSIONS

We have computed the perturbation and hybrid perturbation-Galerkin solutions for the forced Duffing equation and have demonstrated that in general the hybrid method improves the perturbation solution. For values of Ω larger than ω_0=1.5 (above the primary resonance), it appears that the perturbation solution by itself is very close to the numerical solution and hence it is difficult to observe the improvements contributed by the hybrid method. The usefulness of the hybrid method is best demonstrated for values of Ω near the resonance $(0.5 < \Omega < 1.0$ and $1.0 < \Omega < 1.5)$.

We also note that this paper represents one of the first applications of the hybrid method to time dependent problems (see [12]), since previous applications have been largely restricted to boundary value problems ([2], [7], [8]). We feel that the method (or some suitably modified version of it) will be useful for time dependent problems in even broader application areas (e.g. partial differential equations). We are curently investigating some particular applications involving partial differential equations with encouraging initial results.

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