# A ROBUST CONTROL DESIGN TECHNIQUE FOR DISCRETE-TIME SYSTEMS

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**Abstract** A robust state feedback design subject to placement of the closed loop eigenvalues in a prescribed region of unit circle is presented. Quantitative measures of robustness and disturbance rejection are investigated. A stochastic optimization algorithm is used to effect trade-off between the free design parameters and to accomplish all the design criteria. A numerical example is given to illustrate the usefulness of the developed approach.

**چکیده** روشی قوی و انعطاف پذیر جهت طراحی پسخور متغیرهای حالت تحت جایگزاری ویژه مقدارهای سیستم مدار بسته در ناحیهٔ تعریف شده از دایره با شعاع واحد برای سیستم های کنترل گسسته ارائه شده است. مقادیر کمی جهت میزان انعطاف پذیری و وا زنی تزاحم ها مورد بررسی قرار گرفته است. یک الگوریتم جدید بهینه سازی تصادفی به منظور متعادل ساختن پارامترهای آزاد در طراحی و برآورد ساختن الزامات آن به کار گرفته شده است. مثال عددی داده شده در آخر مقاله مفید بودن و برتری روش توسعه داده شده را مصور می کند.

## INTRODUCTION

In designing feedback control systems one is mainly concerned with the transient response and its robustness. The transient requirements usually appear as parametric inequality constraints [1,2]. If the open loop system is completely controllable, then there exists a state feedback gain that places the closed loop poles arbitrarily in the left half plane. In the case of simple output feedback, controllability and observability are not sufficient for the existence of arbitrary eignevalue placement [2]. In the class of dynamic compensation, like the Luenberger observer and the dynamic output feedback controller, it is known that if the open loop system is completely controllable and observable, then the closed loop poles can be located arbitrarily by the proper choice of the gain matrices of an augmented system [3,4].

General analytical constraints on the characteristic polynomial coefficients to solve the stabilization problem have a rich history and are well documented [5,6]. Of particular significance is the set of critical constraints ensuring that eigenvalues remain in a specified region of the complex plane, given allowable perturbations in the system parameters [7].

Theoretical development of discrete time systems has historically lagged similar advances as in the continuous case. Therefore, traditionally attempts have been made to extend the continuous system tools to discrete systems [10,11].

In this paper astate feedback design technique is presented for discrete time systems. The design procedure is based on the assignment of the closed loop eigenvalues in a defined region of unit circle. The control design is robust with respect to system parameter perturbation and provides for disturbance rejection. The Lyapunov method is employed to express the various control objectives related to the time domain performances. A useful quantitative approach to incorporate robustness and disturbances rejection is developed. A stochastic optimization algorithm is used for the selection of feasible control law from the available information. With minor modifications, the result presented here can be extended to continuous case.

## PROBLEM FORMULATION

In this section, the selection of parametrized families of state feedback gains for eigenvalue clustering is considered. Several interesting properties of eigenvectors and Vandermonde matrices are useful here. The major task in achieving spectrum assignment is to derive a procedure establishing a closed-form link between the feedback gain $K$ and the nth order open-loop characteristic polynomial coefficients.

Consider a linear system described by

$$x(k+1)=Ax(k)+Bu(k) \quad (1a)$$
$$u(k)=Kx(k) \quad (1b)$$

where $x(t)\in R^n$, $u(t)\in R^r$, and $A$, $B$, $K$ are constant matrices with compatible dimension. A well known method [8] for determination of state feedback in terms of desired complex eigenvalues, requires satisfying a set of independent linear equations, with attendant redundancy in multi-input problems,

$$K\, col_j[\lambda_{di} I_n - A]^{-1}B=e_j, \quad (2)$$

where $col_j$ indicates the jth column, $\lambda_{di}$ is ith desired eigenvalues, $e_j$ is jth column of the unit matrix $I_r$, $[\lambda_{di}I_n-A]^{-1}$ is the resolvant matrix.

To analyze the resolvant matrix, apply the Cayley-Hamilton Theorem,

$$(zI-A)^{-1}=\sum_{i=1}^{n} g_i(z)A^{i-1}, \quad (3)$$

where $g_i(z)$'s are the unique solutions of:

$$\begin{bmatrix}(z-\lambda_1)^{-1}\\(z-\lambda_2)^{-1}\\ \vdots \\(z-\lambda_n)^{-1}\end{bmatrix}=V^T\begin{bmatrix}g_1(z)\\g_2(z)\\ \vdots \\g_n(z)\end{bmatrix}$$

and $V$ denotes the Vandermonde matrix.

$$V=\begin{bmatrix}1 & & \\ \lambda_1 & \lambda_2 & \lambda_n\\ \vdots & \vdots & \vdots\\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_n^{n-1}\end{bmatrix}$$

Let

$$\Omega=diag(\lambda_i),\ i=1,\cdots,n,$$

then we have

$$\begin{bmatrix}(z-\lambda_1)^{-1}\\(z-\lambda_2)^{-1}\\ \vdots \\(z-\lambda_n)^{-1}\end{bmatrix}=(zI-\Omega)^{-1}\begin{bmatrix}1\\1\\ \vdots \\1\end{bmatrix}$$

Therefore,

$$(zI-\Omega)^{-1}\begin{bmatrix}1\\1\\ \vdots \\1\end{bmatrix}=V^T\begin{bmatrix}g_1(z)\\g_2(z)\\ \vdots \\g_n(z)\end{bmatrix}$$

Premultiply by the inverse Vandermonde matrix,

$$(V^T)^{-1}(zI-\Omega)^{-1}\begin{bmatrix}1\\1\\ \vdots \\1\end{bmatrix}=V^T\begin{bmatrix}g_1(z)\\g_2(z)\\ \vdots \\g_n(z)\end{bmatrix} \quad (4)$$

$$=V^T\begin{bmatrix}0\\ \vdots \\0\end{bmatrix}$$

$$\begin{bmatrix}g_1(z)\\g_2(z)\\ \vdots \\g_n(z)\end{bmatrix}=((V^T)^{-1}(zI-\Omega)^{-1}V^T)\begin{bmatrix}1\\0\\ \vdots \\0\end{bmatrix}$$

$$=(zI-A_{co}^T)^{-1}\begin{bmatrix}1\\0\\ \vdots \\0\end{bmatrix} \quad (5)$$

where $Aco$ is the companion matrix obtained from $\Omega$, that is

$$A_{co}=\begin{bmatrix} 0 & & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & . & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_1 \end{bmatrix}$$

Since

$$(zI-A_{co}^T \begin{bmatrix} 1 \\ 0 \\ \\ 0 \end{bmatrix} = \begin{bmatrix} z^{n-1}+a_1 z^{n-2}+\cdots+a_{n-1} \\ z^{n-2}+a_1 z^{n-3}+\cdots+a_{n-2} \\ \vdots \\ 1 \end{bmatrix}$$

Substituting (5) into (3), we have

$$(zI-A)^{-1}B=\frac{1}{\Delta(z)}\sum_{i=1}^{n}(z^{n-i}+a_1 z^{n-i-1}$$

$$+\cdots+a_{n-i})A^{i-1}B. \qquad (6)$$

Finally, define the open-loop characteristic polynomial as,

$$\Delta(z)=[1,a_1, \quad ,a_n] \begin{bmatrix} z^n \\ z^{n-1} \end{bmatrix} \qquad (7)$$

and the desired closed-loop characteristic polynomial as,

$$\Delta_d(z)=[1,a_{d1}, \quad ,a_{dn}] \begin{bmatrix} z^n \\ z^{n-1} \\ \vdots \\ 1 \end{bmatrix} \qquad (8)$$

Assume that

$$\Delta(\lambda_{di})\neq 0,$$

and

$$\Delta_d(\lambda_{di})=0,$$

resulting in

$$\sum_{i=1}^{n}(a_i-a_{di})\lambda_d^{n-1}=\sum_{i=0}^{n}a_i\lambda_d^{n-i}, a_0=1, \qquad (9)$$

where $\lambda_d$ represents the desired eigenvalues $(\lambda_{d1},\lambda_{d2},\ldots,\lambda_{dn})$. By substitution of (6), and (9) into (2):

$$K\, col_j[\sum_{i=1}^{n}(\lambda_d^{n-i}+a_1\lambda_d^{n-i-1}+\cdot \quad +a_{n-i})A^{i-1}B]$$

$$=\sum_{k=1}^{n}(a_k \quad -a_{dk})\lambda_d^{n-k}\, e_j.$$

If all the desired $\lambda_{di}$ are distinct, it will always be possible to find $n$ linearly independent columns from

$$[\sum_{i=1}^{n}(\lambda_d^{n-i}+a_1\lambda_d^{n-i-1}+\cdot \quad +a_{n-i})A^{i-1}B]$$

where $\lambda_d \in (\lambda_{d1}, \lambda_{d2},\ldots,\lambda_{dn})$. Then matrix $K$ can be found. When repeated poles are desired, a modification is used to find $n$ linearly independent columns [8].

## ROBUSTNESS AND NOISE REJECTION

In this section, we will discuss the performance characteristics due to plant parameter perturbations and input noise. Consider a system discribed by

$$x(k+1)=A_{cl}x(k), x(0)=x_0,$$

where $A_{cl}$ is an asymptotically stable matrix.

An important fact is that if $A_{cl}$ is asymptotically stable, then $P$ is a unique solution to the Lyapunov matrix equation [9],

$$P=A_{cl}^T P\, A_{cl}+0, \qquad (12)$$

and $P=P^T>0$, for any given $Q=Q^T>0$, where '>' denotes that square matrix is positive definite. Suppose that $A_{cl}$ is changed to $A_{cl}+\delta A$, and $P$ is changed to $P+\delta P$, because of the parameter perturbations, then a similar Lyaponov matrix equation is formed as follows,

$$P+\delta P=(A_{cl}+\delta A)^T(P+\delta P)(A_{cl}+\delta A)+Q,$$

Define a performance measure as

$$J=x_0^T P\, x_0, x_0\neq 0, \qquad (14)$$

A quantified measure of the degradation in the performance caused by parameter perturbation in $A_d$ is given as follows,

$$\rho = \max_{\theta^1}\{\frac{J(\theta^1)}{J(\theta^0)};\frac{J(\theta^1)}{J(\theta^0)} \geq 1\}, \qquad (15)$$

where $\theta^0$ represents the nominal plant parameter vector, $\theta^1$ represents the perturbed plant parameter vector. and

$$J(\theta^1) = x_0^T(P+\delta P)x_0. \qquad (16)$$

Rewrite (12) as

$$P=(A_d+\delta A)^T P(A_d+\delta A)+Q, \\ -\delta A^T P A_d - A_d^T P\delta A - \delta A^T P\delta A.$$

Subtracting (17) from (13), we have

$$\delta P=(A_d+\delta A)^T \delta P(A_d+\delta A) \\ +\delta A^T P A_d + A_d^T P\delta A + \delta A^T P\delta A, \qquad (18)$$

(17) multiplied by $(\rho-1)$, then minus (18) yields

$$(\rho-1)P-\delta P=[A_d+\delta A]^T(\rho-1)P-\delta P][A_d+\delta A] \\ +(\rho-1)Q-\rho(\delta A^T P A_d + A_d^T P\delta A \\ +\delta A^T P\delta A), \qquad ($$

Therefore, if

$$(\rho-1)Q-\rho(\delta A^T P A_d + A_d^T P\delta A+\delta A^T P\delta A)>0, \qquad (20)$$

then $(\rho-1)P-\delta P>0$ is the unique solution to (19), which is similar to the result of the continuous time system.

**Theorem:**

If

$$\frac{(\rho-1)^2 Q}{4\rho^2} > (\delta A)^T T Q^{-1} T^T \delta A, \qquad (21)$$

then

$$(\rho-1)Q-\rho((\delta A)^T P A_d + A_d^T P\delta A+(\delta A)^T P\delta A)>0,$$

where

$$T=P(A_d+\delta A/2).$$

**Proof:**

let

$$T=P(A_d+\delta A/2),$$

then we have

$$(\rho-1)Q-\rho((\delta A)^T P A_d + A_d^T P\delta A+(\delta A)^T P\delta A) \\ =(\rho-1)Q-\rho((\delta A)^T T + T^T\delta A). \qquad (22)$$

Let $Q_1^{1/2}$ be the positive symmetric square root of $Q_1=(\rho-1)Q/\rho$, then it is easy to verify that

$$(\frac{1}{\sqrt{2}}Q_1^{1/2}-\sqrt{2}(\delta A)^T T Q_1^{-1/2})(\frac{1}{\sqrt{2}}Q_1^{1/2} \\ -\sqrt{2}Q_1^{-1/2}T^T\delta A)>0, \qquad (23)$$

and then

$$Q_1/2-((\delta A)^T T+T^T\delta A)+2(\delta A)^T T Q_1^{-1}T^T\delta A>0,$$

i.e.

$$Q_1-((\delta A)^T T+T^T\delta A)>Q_1/2-2(\delta A)^T T Q_1^{-1}\cdot T^T\delta A.$$

Therefore, if (21) holds, we have

$$Q_1/2-2(\delta A)^T T Q_1^{-1}T^T\delta A>0.$$

So that

$$Q_1-((\delta A)^T T+T^T\delta A)>0.$$

Recalling $T=P(A_d+\delta A/2)$, we have

$$(\rho\cdot 1)Q-\rho((\delta A)^T P A_d + A_d^T P\delta A+(\delta A)^T P\delta A)>0.$$

The theorem provides a condition to ensure that (20) holds in terms of a generalized norm of $\delta A$.

$\rho$ can be easily used as an index to the performance degradation due to parameter perturbation. While $\delta P$ is primarily a function of the worst possible uncertain plant parameter vector. As we know, however, when the parameters change, the noise rejection property of the system is also affected. In order to develop a measurement of noise rejection, consider a system driven by white noise, i.e.

$$x(k+1)=Ax(k)+\hat{B}\omega(k), \qquad (24)$$

where $\omega(k)$ is a sequence of mutually uncorrelated zero-mean with a constant variance matrix W, i.e. white noise. $x(0)$ has mean $m_0$, and variance matrix $S_0$. The variance matrix of $x(k)$ is defined as

$$S(k)=E([x(k)-m(k)][x(k)-m(k)]^T),$$

and mean $m(k)$ is defined as

$$m(k)=E(x(k)).$$

It can be proved that $S=lim_{k\to\infty}S(k)$ is the unique solution of the Lyapunov equation

$$S=A_d S A_d^T+\hat{B}W\hat{B}^T. \qquad (25)$$

Thus, a measurement of noise rejection can be defined as

$$\sigma=\max_{\delta S}\{\frac{tr((S+\delta S)^2)}{tr(S^2)},\frac{tr((S+S\delta)^2)}{tr(S^2)}\geq 1\}, \qquad (26)$$

where $\delta S$ is a variation of $S$ caused by parameter perturbation. A useful matrix indicator corresponding to (20) is

$$V=(\sigma-1)\hat{B}W\hat{B}^T-\sigma(\delta ASA_d^T+A_d S\delta A^T+\delta AS\delta A^T)>0,$$

The free design parameters in the eigensystem assignment can be chosen such that the system has a better performance as well as a noise rejection property.

## An Optimization Approach

Let free design parameters be denoted by a vector $f\in F\subset R^p$, where F is a set of feasible values for f. We can assume that

$$F=[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]\times\cdots\times[\alpha_P,\beta_P]$$

where $\alpha_i$ and $\beta_i$ are finite. The purpose is to find a set of free parameters such that both $\rho$ and $\sigma$ approach 1.

Define a function

$$g=w_1\rho+w_2\sigma, \qquad (28)$$

where $w_1>0, w_2>0$, and $w_1+w_2=1$[7].

A stochastic optimization algorithm is used to minimize function g with $f\in F$. The procedure is given as follows.

1) Select initial values for f, denote by $f^{(0)}$.

2) Select an initial range $r_i$ for $f_i, i=1,...,p$, denoted by $r^{(0)}$.

3) Set the iteration index $l=1$.

4) Take $m$ sets of p random numbers between -0.5 to 0.5, denoted by $Z_q$, $q=1,...,m$, and for each set $Z_q$ calculate

$$f^{(l)}=f^{(l-1)}+Z_q * r^{(l-1)}.$$

5) If $f_i^{(l)}<\alpha_i$, then $f_i^{(l)}=\alpha_i$, if $f_i^{(l)}>\beta_i$, then $f_i^{(l)}=\beta_i$.

6) For each set, evaluate $\rho$ and $\sigma$. Calculate function $g$.

7) Choose the parameter $f$ which gives the maximum g and denote this as $f_i^{(l)}$.

8) If $g(f_i^{(l)})$ satisfies the termination condition, then stop. Otherwise $l=l+1$.

9) Reduce the range by

$$r^{(l)}=(1-\varepsilon)r^{(l-1)}$$

where $0<\in<1$.

10) Go to step 4 and continue for a predetermined number of iterations.

## Numerical Example

Consider a discretized linearsystem model given by the following,

$$x(k+1)=Ax(t)+Bu(t),$$

where

$$A=\begin{bmatrix}1.0 & 0.0988 & 0.0410 & 0.0010\\ 0.0 & 0.9671 & 0.0721 & 0.0278\\ 0.0 & -0.5768 & 0.4007 & 0.4378\\ 0.0 & 0.2780 & -0.5473 & 0.3738\end{bmatrix}$$

$$B=\begin{bmatrix}0.0003 & 0.0007\\ 0.0103 & 0.0206\\ 0.2780 & 0.3605\\ 0.6965 & -0.1737\end{bmatrix}$$

The design objective is to assign the eigenvalues in the unit circle, and $\lambda_1=0.6277+j0.3935$, $\lambda_2=0.6277-j0.3935$, $\lambda_3\in[0.4274,0.5488]$, $\lambda_4\in[0.3867,0.4274]$. Here, we specify two eigenvalues in terms of subregion of the unit circle instead of exact location. Using the algorithm in the previous section, we can obtain a gain matrix which can assign the eigenvalues in the desired region.

Let $w_1=w_2=0.5$, and $\delta A=\sum_{i=1}^{m}\delta_i E_i$, where $E_i$, are constant matrices determined by the relation among the system uncertanties, $\delta_i$ are uncertain parameters varying in the intervals $[-\varepsilon_i, \varepsilon_i]$.

Select $\lambda_3^{(0)}=0.4490$, $\lambda_4^{(0)}=0.4066$. Then we have the following.

When, $\lambda_3^{(1)}=0.5443$, $\lambda_4^{(1)}=0.4192$.

$$\rho=1.0408, \sigma=1.1304,$$

$$K^{(1)}=\begin{bmatrix}-34.6575 & -7.2471 & -0.4715 & 0.0038 \\ -25.9833 & -8.5993 & -0.9024 & -1.2349\end{bmatrix}$$

When $l=2$, $\lambda_3^{(2)}=0.5488$, $\lambda_4^{(2)}=0.3995$,

$$\rho=1.0268, \sigma=1.1027,$$

$$K^{(2)}=\begin{bmatrix}-34.5926 & -7.2579 & -0.4699 & 0.0037 \\ -25.5079 & -8.8153 & -0.9396 & -1.232\end{bmatrix}$$

Finally, we have when $\lambda_3^{(*)} = 0.4643$, $\lambda_4^{(*)} = 0.4032$ and,

$$\rho=1.0057, \sigma=1.0154,$$

$$K^{(*)}=\begin{bmatrix}-34.7765 & -7.2618 & -0.4743 & 0.0046 \\ -26.2519 & -8.6315 & -0.9087 & -1.2328\end{bmatrix}$$

If $\lambda_1$ and $\lambda_2$ are given by a specification of subregion in the unit circle, we can also apply the algorithm to obtain the suitable gain matrix.

## CONCLUSION

A robust control design method is presented in this paper. The quantitative measures of robustness and disturbance rejection are derived by means of Lyapunov matrix equations. Usually, the requirement of transient performance of a dynamic system can be achieved by assigning the eigenvalues of the closed loop system. However, the robustness and noise rejection requirements are obtained by the selection of free design parameters. A stochastic optimization algorithm is used to make the selection from a set of feasible control gain matrices.

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## REFERENCES

1. J. O'Riely. "Observer for Linear Systems," Academic Press, New York (1984).
2. S. D. Katebi and M. R. Katebi, *Internat. S. System Sci.* 18, 2001 (1987).
3. D. D. Siljak, IEEE *Transaction Automatic Control* AC-34, 674 (1989).
4. C. T. Chen, "Linear Systems Theory and Design," HRW, New York (1986).
5. A. T. Fuller, *J. Math. Anal. Applic.* 23, 115 (1968).
6. S. Gutman and E. Jury. IEEE *Transaction Automatic Control* AC-26, 853 (1981).
7. S. Gutman, *Internat. S. Control* 29, 871 (1979).
8. W. L. Brogan, "Modern Control Theory," Quantum Publisher, New York (1984).
9. R. E. Kalman and J.E. Bertram. *Transaction ASME, Series D, Journal of Basic Engineering* (1960).
10. S. D. Katebi and N.A. Kheir. *Proceedings of the 7th International Conference on Systems Engineering*, University of Nevada, Las Vegas pp. 98-105.
11. T. T. Lee and S. Haran, *Internat. J. System Sci* 18, 117 (1987).