A PARTIALLY OBSERVABLE MARKOVIAN MAINTENANCE PROCESS WITH CONTINUOUS COST FUNCTIONS

M. Aryanezhad

Department of Industrial Engineering
Iran University of Science and Technology
Tehran, Iran

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Abstract In this paper a two-state Markovian maintenance process where the true state is unknown will be considered. The operating cost per period is a continuous random variable which depends on the state of the process. If investigation cost is incurred at the beginning of any period, the system will be returned to the "in-control" state instantaneously. This problem is solved using the average criteria. The method involves exploiting the structure of the problem to develop an algorithm which is shown to be more efficient than the usual dynamic programming approach. Results of extensive tests show the accuracy of this algorithm. In addition, it is shown that if certain condition is satisfied, then it is possible to find the average cost per period by a simple calculation.

چکیده در این مقاله ما یک مسأله برنامه ریزی تعمیر و نگهداری مارکوی دو وضعیتی را که موقعیت واقعی آن مجهول است مورد نظر قرار میدهیم . هزینه کارکردن ماشین در هر پریود یک متغیر تصادفی با تابع چگالی پیوسته است که فرم تابع آن بستگی به وضعیت ماشین دارد . اگر هزینه بازرسی ماشین در ابتدای پریود پرداخت گردد ماشین فورا" به وضعیت " تحت کنترل "انتقال می‌یابد . در این مسأله ما از محک میانگین هزینه هر پریود در یک دوره طویل‌المدت استفاده میکنیم . روش ما منجر به یک الگوریتم فوق‌العاده کاراتر از برنامه‌ریزی پویا میگردد . نتایج تست‌های متعدد نشان میدهد که این الگوریتم از دقت بسیار خوبی نیز برخوردار است . مضافا" براینکه در این مقاله نشان میدهیم که اگر شرایط خاصی برقرار باشد ما میتوانیم میانگین هزینه در هر پریود را بتوسط محاسبات ساده بدست آوریم .

INTRODUCTION

Consider the two state problem of Kaplan [10] (In-Control and Out of-Control) with a Markov chain describing the transition between the two states in successive periods. The operating cost per period is a random variable depending upon the state of the process. In each period there are two actions: continue or investigate. If the investigation cost is incurred at the beginning of any period the system will be returned to the "in-control" states instantaneously. Thus, a decision to delay investigation for one period carries the risk of operating one more period out of control, therefore incurring expenses from the higher cost out-of-control distribution rather than from the lower, cost, in-control distribution. Balanced against this risk is the certain cost of an investigation which might show that the system is still in-control. Examples of this model include inspection/replacement of production machinaries, maintenance of military equipment or communication systems, quality control, and cost variance investigations problem in managerial accounting.

This problem is a partially observable Markovian maintenance process. Such a process is a generalization of a Markovian decision process which permits uncertainty regarding the state of a Markov process and allows state information acquisition. This generalization results in added computational difficulties. In a finite state Markovian decision process, an optimal policy can be expressed in simple tabular form, listing optimal actions for each state. However, in the partially-observable Markovian maintenance process, because of the state uncertainty we will be confronted with an enlarged set of states, in fact a continuum of states.

Monahan [16] in his review of the partially observable Markov decision process, states that efficient computational procedures exist when the planning horizon is finite and short. Less efficient procedures exist for infinite horizon problems.

Here we will reconsider the two-state partially observable Markovian-maintenance process (POMMP) of Kaplan. Several authors have addressed this problem but none has used the approach that we suggest, namely the optimal long run-non-discounted average cost per period.

Kaplan [10] solved this problem by using dynamic programming for a discounted infinite horizon case. However, the convergence of its V_i (The optimal return function with i=1, 2, ... periods to go) is not finite, and in value iteration the repetition of a policy (The same decision rule for V_i as V_{i+1}) does not imply that the policy is optimal. The two-state (POMMP) can also be solved by policy iteration, and the methods are finite when the number of realized cost is finite. Either Brown's [2] method of recursive sets of rules or Sondik's [20] "finitely transient" procedure can be used, and Sondik points their similarity. Magee [14] attempted the solution of the two-state problem allowing cost to be normally distributed. Magee proposed seven plausible rules to be compared through simulation. He did not develop an optimal solution, however, because of the difficulties in doing so. Dittman and Prakash [7] proposed an easily calculated heuristic rule which allows the decision of whether to investigate to the dependence on the most recently observed cost. They investigate the non optimality of their rule in [8].

Buckman and Miller [3] have solved the two-state problem as a regenerative stopping problem, but they modified the parameters of the problem in order to satisfy a monotone condition. Buckman and Miller [3] developed their previous studies for a multiple cost process systems, they assumed each cost process evolves independently. Investigation and correction are assumed to be made for all cost process at once, and investigation decision is based on a vector of probabilities that each cost process is in control.

In this paper the two-state (POMMP) will be solved using the average criteria. Aryanezhad [1] has shown that the average cost per period (ACPP) for an infinite horizon case of the two-state (POMMP) has an optimal policy of the control limit type.

Then he proves that this (ACPP) is a quasi-convex function of the control limit policy. We will use these two main results and a discrete-approximation technique to develop an algorithm using Fibonacci search in order to find the optimal (ACPP) of the (POMMP) model, when the cost functions are continuous. Furthermore it is shown that if certain condition is satisfied, then the (ACPP) will be derived by a simple calculation.

In our algorithm, the new control limit policy will be derived through a searching procedure. We have shown by numerical examples that this searching method is quite accurate and much faster than the dynamic programming approach, used by Dittman and Prakash [8] since it converges to the optimal limit policy from both sides instead of one side. In addition certain other properties are utilized to reduce the search effort.

The paper is organized as follows. In Section 2, the mathematical description of the model is given. Analysis of the model is given in Section 3. In Section 4, we will show how to find the optimal (ACPP) for a special case. In Section 5, we will give a discrete-approximation method in order to find (ACPP) for a given control limit policy. An algorithm using Fibonacci search is given in Section 6. Finally in Section 7, results of extensive tests are presented.

MATHEMATICAL DESCRIPTION OF THE MODEL

We are considering a two state system where state 1 means the system is "in-control" and state 2 means the system is out-of-control. We let q_i be the probability of being in state 1 at the beginning of period i. The density of the expected cost x in period i will be $q_i f_1(x) + (1-q_i)f_2(x)$ where $f_1(x)$ and $f_2(x)$ are the probability densities of cost when we are in state 1 or 2 respectively. Presumably $f_1(x)$ has most of its probability at low costs and $f_2(x)$ has most of its probability at higher costs. We let m_1 and m_2 be the means of the two distributions.

When in state 1, there is a probability g of remaining in state 1 and probability $(1-g)$ of going to state 2. It is assumed the move to state 2 takes place late enough in any given period so that reported costs are determined by the state the system is in at the beginning of the period. The cost in the given period and the state the system goes to in the next period are assumed to be conditionally independent given the state at the beginning of the period. Once the system is in state 2, it will remain there until corrective action is taken. Therefore, the system can be represented by a two-state Markov process whose one step transition matrix is:

$$P = \begin{bmatrix} g & 1-g \\ 0 & 1 \end{bmatrix}$$

Investigation and repair are synonymous in this model. Either action may be taken at the beginning of any period for a cost of K and the system will be in state 1 instantaneously. The cost K is incurred even though the system may have already been in state 1 and no correction was needed.

The probability that the system is in state 1 will be determined because the actual state of the system cannot be known except through investigation. Since q_i is the probability of being in state 1 at the beginning of period i, we can find q_{i+1} given that the cost is x by using Bayes formula:

$$q_{i+1/x} = \frac{P(\text{x/we are in state 1})P(\text{being in state 1})}{P(\text{The cost is x})}$$

$$= \frac{g f_1(x)\, q_i}{q_i f_1(x) + (1-q_i)\, f_2(x)} \qquad (1)$$

In state q_i two decisions are available, decision 1 which is to do nothing and decision 2 which is to investigate and correct if necessary. Therefore, the one period cost and state transition functions will be as follow:

$$C(q_i, 1) = q_i m_1 + (1-q_i)m_2;$$

$$q_{i+1} = \frac{g\, q_i f_1(x)}{q_i f_1(x) + (1-q_i) f_2(x)}$$

$$C(q_i, 2) = K + g m_1 + (1-g)m_2;$$

$$q_{i+1} = \frac{g^2 f_1(x)}{g f_1(x) + (1-g) f_2(x)}$$

The objective is to determine a decision rule which minimizes λ, the average cost per period (ACPP) for an infinite planning horizon.

Let E[C] and E[T] be the total expected cost and the total expected time until stopping respectively. Then the (ACPP) turns out to be

$$\lambda = \frac{E[C]}{E[T]} \tag{2}$$

In order to simplify the calculations of optimal λ^* we need to exploit the structure of the problem. This will be the subject of the following section.

MODEL ANALYSIS

Let q_i be the state of the system at period i (initially $q_0=g$) Then the one step transition probability P(U, V) is the probability of going from state U to state V, where $U\epsilon S$ and $V\epsilon S$ and $S= (o,g)$; $g<1$. Hence we will have a Markov chain with continuous state. Aryanezhad [1] has proved that the optimal policy is of the control limit type. In the other words, suppose $\overline{q}$ is the stopping level, then we won't inspect the system unless $q_i < \overline{q}$. Therefore, by using the result of section (VI.II) of Feller [9], the probability of going from state $q_0=g$ to state q after n steps will be as follows:

$$P^n(g, q) = \int_{\overline{q}}^{g} \int_{\overline{q}}^{g} \cdots \int_{\overline{q}}^{g} P(g, q_1)\, P(q_1, q_2) \cdots P(q_{n-1}, q)\, dq_1 dq_2 \cdots dq_{n-1} \tag{3}$$

then by Cinlar [5] we have

$$E[T] = \int_{\overline{q}}^{g} \sum_{n=0}^{\infty} P^n(g, u)\, du \tag{4}$$

Calculation of E[T] by (4) is not so simple. In the following section it will be shown that if certain condition is satisfied (i.e. $f_1(x)$ is uniformly distributed in x), then it will be possible to find an explicit formula for λ as a function of $\overline{q}$. In section 5 we will propose a discrete approximation method in order to find λ for a given stopping level $\overline{q}$.

CALCULATION OF λ^* (SPECIAL CASE)

Here, we would like to show that if a certain condition is satisfied for $f_1(x)$, then it is possible to find an explicit formula for λ as a function of $\overline{q}$. Before doing it, let us prove the following lemma and theorem for our continuous state Markov chain.

LEMMA 1. Suppose we have a Markov chain with continuous and transient states $U\epsilon S$, $V\epsilon S$ and $S=\{\overline{q}, g\}$, $0<\overline{q}<g<1$. Let P (U, V) the one step transition probability to be a separable function of U and V i.e. $P(U, V)=\eta(U)\theta(V)$. Suppose $\eta(U)$ and $\theta(V)$ are positive functions of U and V respectively. Let us assume for each value of $U\epsilon S$, V can take all values of S. Suppose S is an irreducible set. Then the n step transition probability will be:

$$P^n(U, V) = \eta(U)\theta(V)\left[\int_{\overline{q}}^{g} \eta(t)\,\theta(t)\, dt\right]^{n-1} \tag{5}$$

PROOF. The result can be proved by induction For n=1 we will obtain the one step transition probability $P(U, V)=\eta(U)\theta(V)$. So it is true, for n=1.

Now, suppose it is true for n−1 or

$$P^{n-1}(U, V)=\eta(U)\theta(V)\left[\int_{\overline{q}}^{g} \eta(t)\theta(t)dt\right]^{n-2}$$

So,

$$P^n(U, V) = \int_{\overline{q}}^{g} P^{n-1}(U, t)P(t, V)dt$$

$$= \int_{\overline{q}}^{g} \eta(U)\theta(t)\left[\int_{\overline{q}}^{g} \eta(w)\theta(w)dw\right]^{n-2}\eta(t)\theta(V)dt$$

$$= \eta(U)\theta(V)\left[\int_{\overline{q}}^{g} \eta(t)\theta(t)dt\right]^{n-1}$$

O. E. D.

THEOREM 1. Suppose all conditions in Lemma 1 are satisfied. Then $E_g[t_u]$ the expected number of visits to state u starting, from g will be:

$$E[t_u] = 1 + \frac{\eta(g)\,\theta(u)}{} \quad (6)$$

PROOF. At period zero, we are in state g with probability 1.

By Lemma 1, we are in state u, starting from g after n period with probability:

$$P^n(g,u) = \eta(g)\,\theta(u)\,[\int_{\bar{q}}^{g} \eta(t)\theta(t)\,dt]^{n-1}$$

Therefore $E_g[t_u]$ will be the sum of all these probability when $n \to \infty$.

$$E_g[t_u] = 1 + \sum_{n=1}^{\infty} \eta(g)\,\theta(u)\,[\int_{\bar{q}}^{g} \eta(t)\,\theta(t)\,dt]^{n-1}$$

Since we are interested in $\bar{q} > 0$.

Then this is a geometric series with, rate $\int_{\bar{q}}^{g} \eta(t)\theta(t)\,dt < 1$. Finally:

$$E_g[t_u] = 1 + \frac{\eta(g)\theta(u)}{1 - \int_{\bar{q}}^{g} \eta(t)\theta(t)dt}$$

O. E. D.

Now let us return to our model to see for what type of $f_i(x)$; i=1, 2 we can use the result of Theorem 1.

By Bayes equation when the cost is x and we are in state q we have

$$q = \frac{gqf_1(x)}{qf_1(x) + (1-q)\,f_2(x)} \quad (1')$$

where q and q′ belong to S, and $S = \{o, g\}$.

At each state q the cost x will lead us to state q′ with probability:

$$P(q, q') = q\,f_1(x) + (1-q)\,f_2(x) \quad (7)$$

Using equation (1) we have

$$P(q, q') = \frac{gqf_1(x)}{q'} \quad (8)$$

Suppose we know q and q′, from equation (1′) and equation (7) we find x and then use its value in equation (8). After this step if P(q,q′) satisfies the separability condition in Theorem 1, then we can easily find E[C], E[T] and $\lambda(\bar{q})$. Now if the solution of $\frac{d\lambda(\bar{q})}{d\bar{q}} = 0$ implied $q = q^* \epsilon S$ then $\lambda^* q^*$ will be the optimal solution.

As an example, let us assume that $f_1(x)$ is uniformly distributed. That is

$$f_1(x) \begin{cases} \frac{1}{a} & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

then by equation (8) we see that

$$P(q, q') = \frac{gq}{aq'}$$

Obviously P(q, q′ is a separable function of q and q′. Then $\eta(q) = q$ and $\theta(q') = \frac{1}{q'}$. Using Theorem 1 we will have

$$E_g[t_{q'}] = 1 + \frac{g^2/aq'}{1 - g(g-\bar{q})/a}$$

Since for each value of $q \epsilon (\bar{q}, g) = S$, q will assume all values belong to S. Then we can write

$$E[T] = \int_{\bar{q}}^{g} E_g[t_{q'}]\,dq'$$

$$= (g - \bar{q}) + \frac{(g^2/a)\ln(g/\bar{q})}{1 - g(g-\bar{q})/a}$$

$$E[C] = \int_{\bar{q}}^{g} C(q')\,E_g[t_{q'}]\,dq$$

$$= \int_{\bar{q}}^{g} [q'm_1 + (1-q')\,m_2] \times$$

$$[1+\frac{g^2/aq'}{1-g(g-\overline{q})/a}]\,dq'$$

Therefore $\lambda=\frac{E[C]}{E[T]}$ will be function of $\overline{q}$. Suppose by solving $d\lambda(\overline{q})/d\overline{q}=0$ we obtain $q=q^*$. Now if $q^*\epsilon S$ then $\lambda^*=\lambda(q^*)$ will be the optimal average cost per period.

CALCULATION OF $\lambda(\overline{q})$

Here we will show how to find λ for a given stopping policy $\overline{q}$. The main problem here is the calculation of the equation (4) or more actually the construction of the matrix P(U, V). This P(U, V) is a continous state Markov chain. Construction of the matrix P(U, V) is quite possible if we use the following four devices of a discrete-approximation. Then it is shown that this approximation is very accurate.

1. Neglect that part of the distribution which has very small probability (i.e. out of $\mu\pm4.5\sigma$ in normal distributions).
2. Choose a good subinterval of integration (e.g., $.01\sigma$ in normal distributions).
3. Convert the continuous state q into discrete states with an interval of .01.
4. Use linear interpolation in order to find the matrix p.

By using the first two ideas our continuous cost function will be changed to a discrete one with a finite number of values. However, we have to normalize these new distributions. Since at each state q_i, there would be many costs, therefore the number of states will be extremely large. The third idea will remove this difficulty.

Now let us construct the matrix P. Suppose that the stopping level is q. Then the state space will be(g, g−0.01, g−0.02, ..., $\overline{q}$) Let $n=(g-q)/0.01$. Let us rename our state space to be $(q_0, q_1, ..., q_n)$ such that $q_i=g-0.01i$, i=0, 1, 2, ..., n. Therefore the matrix P will be (n+1) x (n+1) matrix. Suppose we are in state q_i, i=0, 1, ..., n and the cost is x. By using Bayes equation (1) we have

$$q=\frac{gq_if_1(x)}{q_if_1(x)+(1-q_i)f_2(x)}$$

If $q<\overline{q}$ we have nothing to calculate. Suppose $q_{j+1}<q<q_j$, j=0, 1, ..., n. By using the 4th idea and the cost distribution in state q_i we will have

$$P(q_i, q_j)=\frac{q-q_{j+1}}{0.01}\,[q_if_1(x)+(1-q_i)f_2(x)]$$

$$P(q_i,q_j+1)=\frac{q_j-q}{0.01}\,[q_if_1(x)+(1-q_i)f_2(x)]$$

Since we are interested in $\overline{q}<0$ then $\sum_j P_{ij}<1$. Therefore $(I-P)^{-1}$ always exists. The expected number of visits to each state-starting from state q_i will be the i+1th row of $(I-P)^{-1}$, Cinlar, E[5]. Since we start form state $q_0=g$ so our interested row will be the first row of $(I-P)^{-1}$. Let us denote the first row of $(I-P)^{-1}$ by $(I-P)^{-1}_{oi}$; i=0, 1, 2, ..., n. Therefore, the expected time and the expected cost and cost and $\lambda(\overline{q})$ will be as follows:

$$E[T]=\sum_{i=o}^{n}\left\{(I-P)^{-1}\right\}_{oi} \tag{9}$$

$$E[C]=K+\sum_{j=0}^{n}\left\{(I-P)^{-1}_{oj}\right\}C(q_j)$$

where

$$C(q_j)=q_jm_1+(1-q_j)m_2$$

Then

$$\lambda(\overline{q})=\frac{E[C]}{E[T]} \tag{12}$$

Aryanezhad [1] has shown that the function λ is a uniminal function of the stopping level.

That is if q_1, q_2 and q_3 are three stopping levels such that $q_1 > q_2 > q_3$ then.

$$\lambda(q_2) < \text{Max}\ [\lambda(q_1), \lambda(q_3)]$$

Therefore if we let the distance of uncertainty equal to .01 in the Fibonacci search we can continue our calculation by using the following algorithm.

ALGORITHM

Calculation of $\lambda(\bar{q})$ in section 5 enables us to give an efficient algorithm to solve the two-state (POMMP) when the cost functions are continuous.

Before giving the algorithm, let us recall the elements of the Fibonacci search, Luenberger, D. [13]

$d_1 = R - L$ $(R=g,\ L=0)$, the initial width of uncertainty.

d_M=width of uncertainty after M measurements.

F_M= the integer number of Fibonacci sequence generated by the recurrence relation $F_N = F_{N-1} + F_{N-2}$; $F_0 = F_1 = 1$.

Then, if a total of N measurements are to be made, we have

$$d_M = \left(\frac{F_{N-M+1}}{F_N}\right) d_1$$

Now we are ready to see the algorithm.

STEP0. Choose the number of measurement points N in the Fibonacci search such that the final distance of uncertainty be equal to .01.

STEP 1. Find d_M (initially M=2). Let

$$\bar{q}_1 = L + d_M$$

$$\bar{q}_2 = R - d_M$$

Find $\lambda(\bar{q}_1)$ and $\lambda(\bar{q}_2)$ by applying the results in section 5.

STEP 2. If $\lambda(\bar{q}_1) = \lambda(\bar{q}_2)$ GO TO STEP 4. Otherwise GO STEP (3).

STEP 3. M=M + 1. If $\lambda(\bar{q}_1)\quad \lambda(\bar{q}_2)$ then $R=\bar{q}_1$ Otherwise $L=\bar{q}_2$. If $M < N$ GO TO STEP 1. Otherwise GO TO STEP (4).

STEP 4. STOP. By quasi-convexity of λ the minimum λ must lie in this final distance of uncertainly (L. R).

NUMERICAL EXAMPLES

We have used Dittman and Prakash's [8] basic examples. In those examples f_1 is $N(100, \sigma_1^2)$ and f_2 is $N(120, \sigma_2^2)$ where σ_1 and σ_2 can range over five values: (5, 10, 15, 20, 30) giving a total of 25 combinations. K=20 and g = .90.

These examples have been solved by the algorithm in section 6. Note that we let the subinterval of integration in fact 2 be equal to .01 min (σ_1, σ_2).

In Table 1 we have presented the detailed calculation of the first case ($\sigma_1 = \sigma_2 = 5$). The final results of all combinations are given in Table 2.

Dittman and Prakash [8] have solved these examples by using the dynamic programming procedure formulated by Kaplan [10].

However, they tabulated $m_2 - \lambda^*$ which they call cost saving, in Table 1 of their paper. We can readily see that the results are the same. It should be mentioned that, in order to derive the results in Table 2, they used more than 3000 seconds of computer time, while it took us less than 340 seconds with IBM 360/91kk. Therefore we can state that while this algorithm is quite accurate, it is much faster than dynamic programming approach.

Table 1

$\sigma_1=\sigma_2=5$		
Region of incertainty in Fibonacci search	$\overline{q}$	$\lambda(\overline{q})$
(0,.90)	.56	194.232697
(0,.90)	.34	104.218552
(0,.56)	.22	104.240692
(.22, .56)	.43	104.216278
(.22, .56)	.35	104.217819
(.35, .56)	.48	104.219757
(.35, .48)	.40	104.216049
(.35, .43)	.38	104.216202
(.38, .43)	.41*	104.215988*
(.40, .43)	.42	104.216064

(*) means the optimal solution

The number of iterations for all combinations is ten.

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Table 2.Data in each cell: λ^*, the optimal average cost per period
q^*, the optimal policy

σ_1 \ σ_2	5	10	15	20	30
5	104.216 .41	104.547 .43	104.76 .45	104.816 .46	104.707 .45
10	105.251 .40	105.542 .44	105.7 .48	105.753 .49	105.543 .49
15	105.885 .41	106.427 .44	106.59 .48	106.623 .51	106.412 .52
20	105.981 .41	106.728 .44	107.153 .47	107.289 .50	107.176 .54
30	105.885 .41	106.651 .44	107.279 .46	107.73 .48	108.09 .52

(Note: q^* and λ^* have been rounded off to the nearest second and third number after decimal point, respectively).

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