STIFFNESS MATRICES FOR AXIAL AND BENDING DEFORMATIONS OF NON–PRISMATIC BEAMS WITH LINEARLY VARYING THICKNESS

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Abstract Siffness matrices for axial and bending deformations of a beam having a rectangular cross sectional area of constant width and linearly varying thickness are developed. A consistant load vector for a uniformly distributed lateral load is also calulated, using the principal of potential energy. The matrices are used to obtain numerical results for a variety of beams with non-uniform thickness to show that acceptable degrees of accuracy can be obtained. A comparison of results given by other finite element solutions is made to show the effectiveness of the derived stiffness matrices.

چکیده در این مقاله ماتریسهای سختی برای تغییر فرمهای محوری و خمشی یک تیر با مقطع مربع مستطیل که پهنای آن ثابت و ضخامتش بطور خطی تغییر می‌کند بدست آورده شده است . و نیز با استفاده از اصل انرژی پتانسیل برای بار گسترده یکنواخت جانبی گاما ، بردار نیروهای گرمای محاسبه شده است . این ماتریسها برای انواع گوناگونی از تیرهای با مقطع متغیر بکار رفته و نتایج عددی حاصل نشان داده‌اند که دقت قابل قبولی را میتوان بدست آورد . همچنین برای آنکه موثر بودن این ماتریسهای سختی بیان شود نتایج با جوابهای حاصل از عناصر محدود دیگری نیز مقایسه شده‌اند .

INTRODUCTION

The development of methods of analysis for framed skeletal structures spread over a period of more than five decades. As more complex skeletal configurations were introduced, new and special analytical methods were developed. Such methods as virtual work. Slope deflection, moment distribution, strain energy and complementary energy, to name a few, were developed for particular applications and for time-saving exercises, since slide rulers and, later, desk calculators were the only computational aids available. With the advent of electronic digital computers, the emphasis turned away from the previous forms of specialization into a method of generalization. The fundamentals of formulating conventional methods in matrix notations were easily illustrated and widely accepted. The use of the resulting matrix methods for the elastic linear analysis for skeletal structures became a popular exercise partly because of the existance of exact relationships between generalized forces (force and bending moment) and corresponding generalized displacements (deflection and rotation) and also because of the simplicity of establishing a standard computational procedure for satisfying equilibrium, compatibility and for the solution of the resulting linear simultaneous algebraic equations. On the other hand, for two dimensional plated types of structures, the exact relationship between generalized forces and corresponding displacements are known only for special simple cases of loading and geometry. A unified approach was, therefore, not generally possible until the advent of the finite element method of analysis. In this method, the continum with an infinite degree of freedom is approximated by a number of finite elements, each with a specified finite number of degrees of freedom.

The dividing line between the matrix and the finite element method cannot be drawn as precisely as it appears to have been suggested previously since the analysis of some skeletal structures also requires the introduction of some approximations concerning geometry and loading. For example, the analysis for bending of non-prismatic beams having a variation of thickness along the length is usually carried out by assuming the beam to be made up of a number of finite lengths, each having a different constant thickness. Such a solution will contain discontinuities of internal generalized forces at the junctions between adjacent elements. To eliminate these discontinuities, an account of the variation of cross-sectional properties along the element must be taken into consideration in the derivation of the stiffness matrix.

If such an element is based on the same assumed cubic polynomial displacement field as for the constant thickness element for bending of beams, it will lead to an assumed linear variation of curvature along the length of the element, and the distribution of the bending moment along the element becomes complex and does not satisfy equilibrium. Instead, the element can be based on a curvature (strain) distribution resulting from a linear variation of the bending moment. The resulting stiffness matrix will be for an equilibrium element, in accordance with the strain based elements suggested by Sabir et al [1-6] for general plane elasticity problems, arches and shells.

In the present paper two such elements are developed: one for axially loaded and one for laterally loaded non-prismatic beam members having a rectangualr cross section of constant width and a linearly varying thickness. The stiffness matrices will be exact provided that the loads are applied at the ends or the nodes of the element. In the case of the bending element, a consistent load vector is also obtained for a uniformly distributed lateral load by using the principal of potential energy. The bending element is used in the present paper to obtain solutions for statically determinate tapered beam problems. The results are compared with those obtained by the use of constant thickness elements and the load lumping process. In this way the effect of these approximations on the convergence of the results is studied.

DISPLACEMENT FIELDS FOR LINEARLY VARYING THICKNESS

In this section, the displacement fields for a beam with a rectangular cross-sectional area, having a constant width b and having a linearly varying thickness from t_1 at one end to t_2 at the other, are derived for the two cases of axial and lateral deformations.

1. Axially Loaded Tapered Element

Consider the element shown in Figure (1). If its length is and the axial coordinate x is measured along the element, the axial strain ϵ_x will be given by

$$\epsilon_x = \frac{du}{dx} = \frac{P_x}{EA_x} = \frac{P_x}{Ebt_x} \qquad (1)$$

where u is the axial displacement.

E is the Young's modulus, P_x, t_x and A_x are the load, thickness and cross-sectional area at x respectively, and b is the constant width. For the element to be in equilibrium, p_x must be constant everywhere along the beam and t_x will be given by

$$t_x = t_1 + \frac{t_2 - t_1}{l} x = t_1 + \frac{a}{x} x \qquad (2)$$

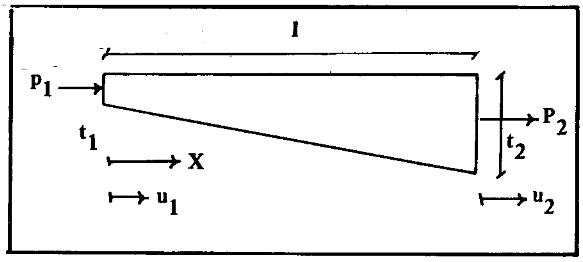

Figure 1. Axially loaded tapered element.

where ℓ is the length of the element and $a = t_2 - t_1$. Hence from (1) and (2)

$$\frac{du}{dx} = \frac{P_x}{Eb(t_1 + \frac{a}{x} x)} \qquad (3)$$

$$\frac{du}{dx} = \frac{A_2}{t_1 + \frac{a}{x} x} \qquad (4)$$

where A_2 is a constant

Integrating (4) will result into an expression for the displacement field for this element, hence

$$u = A_1 + \frac{A_2 l}{d} \ln (t_1 + \frac{ax}{}) \qquad (5)$$

This expression is in terms of two constants A_1 and A_2. A_1 represents the rigid body mode of displacement and A_2 is due to straining. The element is thus to have two degrees of freedom, namely, the axial displacement at each and or node.

2. Laterally Loaded Tapered Element

Consider the element shown in Figure (2). If W denotes the lateral deflection due to the applications of nodal shearing forces and bending moments, and the element is to be in equilibrium, the variation of the bending moment along the element must be linear and will be given by

$$M_x = -M_1 + F_{x_1} \qquad (6)$$

where F_1 and M_1 are the applied shear force and bending moment at node 1.

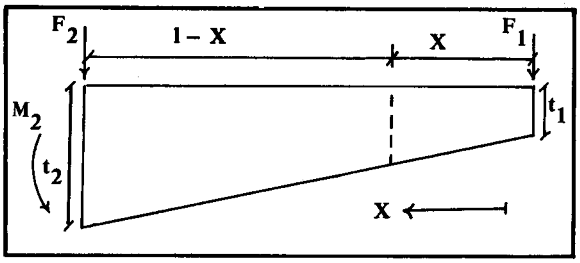

Figure 2. Laterally loaded tapered element.

M_x will be given from the simple bending theory by

$$EI_x \frac{d^2w}{dx^2} = M_x \qquad (7)$$

where I_x is the second moment of area of the cross-section and is given by $Ebt^3x / 12$. Hence

$$\frac{d^2w}{dx^2} = \frac{-M_1 + F_1 x}{\frac{Eb}{12}(t_1 + \frac{ax}{l})^3} \qquad (8)$$

or

$$\frac{d_2w}{dx^2} = \frac{A_3 + A_4 x}{(t_1 + \frac{ax}{})} \qquad 9)$$

where A_3 and A_4 are constants. Equation (9) can be integrated once to give the variation of the slope of the deflected shape along the beam. Hence

$$\frac{dw}{dx} = A_2 + \frac{A_3 l}{2a(t_1 + \frac{ax}{l})^2} + \frac{A_4 l^2 t_1}{2a^2(t_1 + \frac{ax}{l})^2}$$

$$\frac{A_4 l^2}{a^2 (t_1 + \frac{ax}{})} \qquad (10)$$

and the final displacement field, w, for the element, will be given by

$$w = A_1 + A_2 x + \frac{A_3 l^2}{2a^2(t_1 + \frac{ax}{l})} - \frac{A_4 l^3 t_1}{2a^3(t_1 + \frac{ax}{l})}$$

$$\frac{A_4 l^3}{a^3} \ln (t_1 + \frac{ax}{l}) \qquad (11)$$

Expression (11) contains four independent constants. A_1 and A_2 represent the two strain-free rigid body modes of displacements, and A_3 and A_4 are due to straining of the element. The element is therefore to have four degrees of freedom, namely, w and $\frac{dw}{dx}$ at each node.

STIFFNESS MATRICES FOR TAPERED ELEMENT

Having obtained the displacement fields for the axially and laterally loaded tapered elements, the stiffness matrices are obtained from the well-known expression (see Zienkiewicz [7]).

$$K = [C^{-1}]^T \left\{ \iint [B]^T [D] \ [B] \, dv \right\} [C]^{-1} \qquad (12)$$

where [K], [C], [B] and [D] are the stiffness, transformation, strain and rigidity matrices.

1. Stiffness Matrix for Axilly Loaded Tapered Element

In this case, the strain matrix [B] will be given by [4] and in matrix notations

$$[B] = [\,0 \quad 1/(t_1 + \frac{ax}{l})] \qquad (13$$

and $D = EA_x = Eb\,(t_1 + \frac{ax}{l}$

The bracketted integral part of equation (12) for an element of length will be given by

$$\int_0^1 [B]^T [D] [B] \, dx = \begin{bmatrix} 0 & 0 \\ 0 & \frac{Ebt}{a} \ln \frac{t_2}{t_1} \end{bmatrix} \qquad (14)$$

The transformation matrix [C] and its inverse are given in Appendix I. When these are introduced into (12), the resulting stiffness matirx [K] will be in the explicit form given below

$$[K] = \frac{Eba}{\ln \frac{t_2}{t_2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (15)$$

It is of interest to note that the above matrix will reduce to the well-known stiffness matrix for a constant thickness element since limit

2. Stiffness Matrix for Laterally Loaded Tapered Element

In this case, [B] is given by [9], hence

$$[B] = [\,0 \;\; 0 \;\; 1/(t_1 + \frac{ax}{l})^3 \; x/(t_1 + \frac{ax}{l})^3] \qquad (16)$$

$$\text{and } D = EI_x = Eb\,(t_1 + \frac{ax}{l})^3/12 \qquad (17)$$

When (16) and (17) are substituted into (12), the necessary matrix multiplications are carried out and the integrations are performed, we obtain

$$\int_0^1 [B]^T [D] [B] \, dx = \frac{Eb}{12} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_1 & h_2 \\ 0 & 0 & h_2 & h_3 \end{bmatrix} \qquad (18)$$

where

$h_1 = l\,(t_2 + t_1) \quad 2t_1^2 t_2^2$

$h_2 = l_2/2t_1 t_2^2$

$$h_3 = \left\{ l^3/(t_2 - t_1)^2 \right\} \left\{ \ln \frac{t_2}{t_1} / (t_2 - t_1) \right.$$
$$\left. (3t_2 - t_1) / 2t_2^2 \right\}$$

The transformation matrix for this element is given in appendix II. The necessary matrix inversions and multiplications are then carried out to obtain the explicit form of the stiffness matrix [K] for bending. The non zero terms of this stiffness matrix are also given explicity in Appendix II.

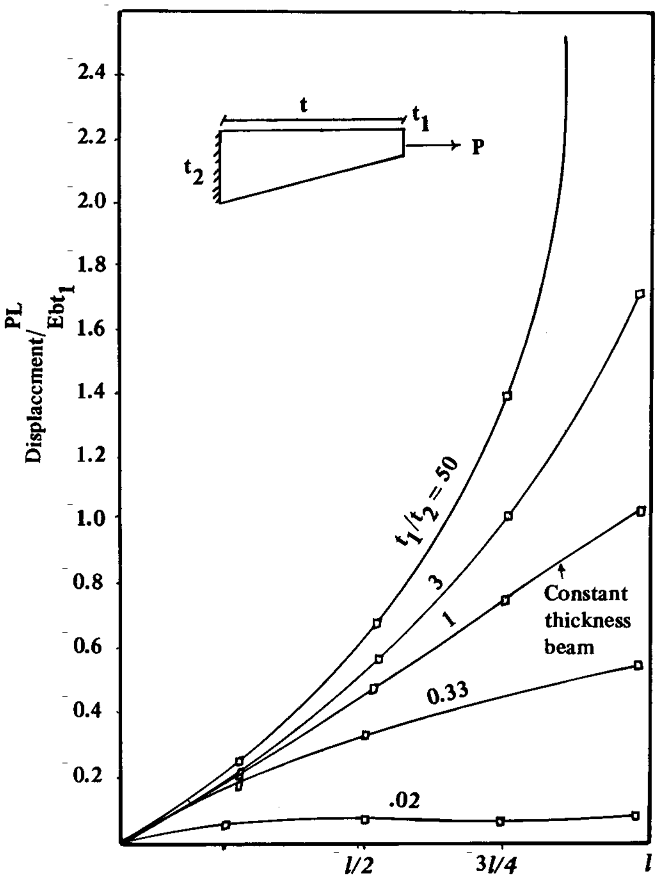

Figure 3. Variation of axial displacement along the beam for a series of constant Values of t_1/t_2.

CONSISTENT LOAD VECTOR

A consistent load vector for a tapered element subjected to a uniform lateral load of q per unit length is calculated using the principal of potential energy, i.e. by equating the work done by the applied load on the deformation of the element to the work done by the equivalent nodal forces on the nodal displacements. Hence

$$\delta_e{}^T \{P\} = \int_0^1 W^T q\, dx \qquad (19)$$

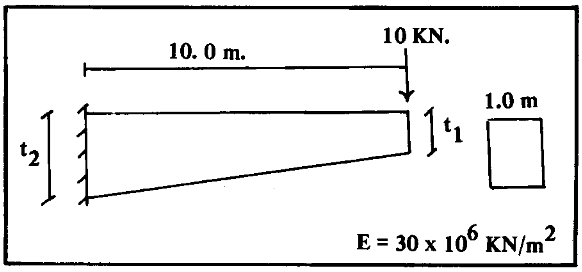

Figure 4. A cantilever tapered beam with point load.

where [P] is the consistent load vector, $[\delta_e]$ are the nodal displacements (deflection and rotation) and W is the displacement field given by equation (11). It can be shown that equation (19) will yield

$$\{P\} = [C^{-1}]^T q \begin{bmatrix} l \\ l^2 12 \\ \frac{l^3}{2a^3} \ln \frac{t^2}{t^1} \\ -\frac{l^3}{a^3}\left\{\frac{1}{a}(t_2 \ln t_2 - t_1 \ln t_1) + 1\right\} \end{bmatrix}$$

where [C] is given in Appendix II.

NUMERICAL RESULTS

All the results given in the present paper were obtained by the use of the NODAL solution routine, Sabir [8]. As a primary example, the stiffness matrix as given in expression (15), for the case of axially loaded tapered member, is used to obtain a solution to a beam fixed at one end and loaded by a tensile axial force P at the other end. Figure (3) gives the distribution of the axial displacement u along the length of the beam for a series of values of t_1/t_2 ranging from 0.02 to 50. Since the stiffness matrix is exact for this case of loading, the entire beam is idealised by element. The results show that linear behaviour is exhibited only for the case of t_1/t_2, and that the degree of nonlinearity increases with the taper of the beam. It is of interest to note that, if a finite element based on an assumed linear displacement field had been used for the analysis of such tapered beams, it would have been necessary to idealise the beam by several constant thickness elements: The effectiveness of the tapered beam element in bending is demonstrated by the analysis of the cantilever beam, as shown in Figure (4). The ratios of the thickness at both ends $t_1:t_2$ were taken to be as those given in Table 1. In all cases, exact results for deflection and the rotation at the loaded end were, as expected, obtained by idealising the beam by one tapered element. These results also show that no numerical difficulties existed when $t_1:t_2$ was made to be nearly one (7.99:8).

The same problem is analysed by idealising the beam by a series of constant thickness beam elements, and the results for the case $t_1:t_2$=48 are given in Table 2. In all cases, the beam is divided into elements of equal lengths.

The table shows that the beam needs to be divided into more than six elements to ensure results with less than 1% error.

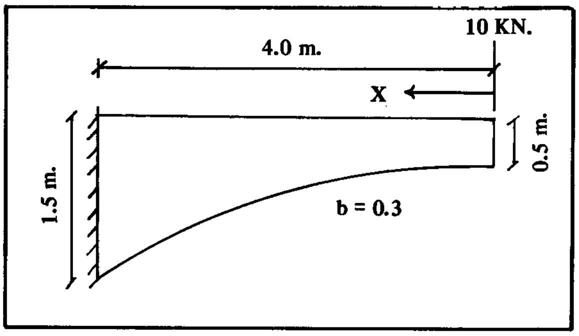

Figure 5. A cantilever beam with non-linearly varying depth

In many practical cases, for example in girder bridges, the thickness of the beams may vary in such a way that their bottom surface follows a smooth curve. We therefore analysed. the beam shown in Figure (5). The thickness of the beam was taken to vary according to

Table 1. Deflection and slope at free and for several values of $t_1:t_2$

$t_1:t_2$	Deflection at free end (mm)	Slope at free end
4:8	42.60	–0.781
7:8	28.77	–0.477
7.99:8	26.05	–0.390
8:8	26.04	–0.3906
8:7	30.06	010.444
8:4	120.70	011.560

Table 2. Convergence of deflection and slope at free end.

No. of ele-ment	deflection at free and (mm)	% error	slope at free end	% error
1	61.73	44.9	0.926	18.6
2	51.40	20.7	0.885	13.3
3	44.65	4.8	0.809	3.6
4	43.74	2.7	0.797	2.0
5	43.32	1.7		1.3
6	43.09	1.2	0.788	0.9
8	42.88	0.7	0.785	0.5

the following equation.
the beam was taken to vary according to the following equation.

$$t_x = 0.5 + \frac{x_2}{16}$$

The beam was analysed by dividing it into 2, 4, 6, 8 and 16 tapered and constant thickness elements. The results for the shearing stresses at the neutral axis and the bending stress at the extreme fibres from the neutral axis were calculated by first multiplying the nodal generalised displacements by the element stiffness matrix to obtain the internal generalised forces. These calculated values of the shearing forces and bending moments were then used to calculate the corresponding stresses in the usual way.

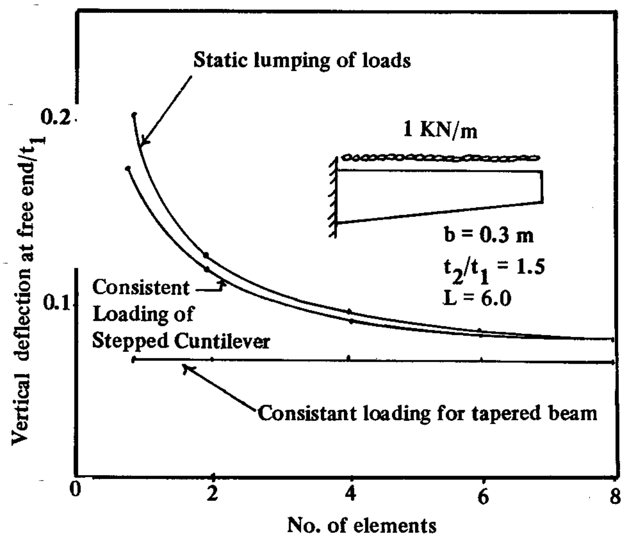

Figure 8. Comarison of results for deflection at free end for different lateral load idealisations.

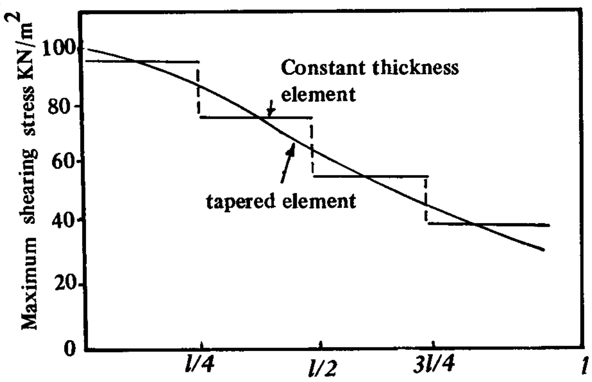

Figure 6. Distribution of maximum shearing stress along the beam.

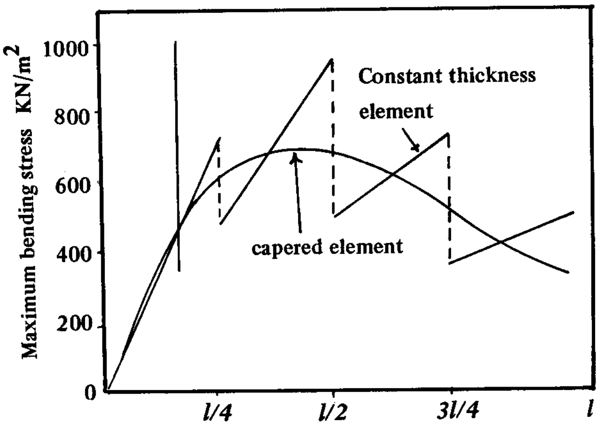

Figure 7. Distribution of maximum bending stress along the beam.

Figure 6 gives a sample of such results for distribution of the maximum shearing stress, along the beam when the beam is idealised by four elements of equal length. This figure shows the dependence of the results obtained from the constant thickness element on the discontinuities in order to obtain the variation along the beam, while the results obtained, using the tapered elements, follow a smoot curve. Figure. 7 gives a sample of such results for the same idealisation. Again we see that a considrable amount of discontinuities exist for the solution obtained by using elements with constant thickness.

From the comprehensive results obtained for this problems, it can be concluded that acceptable converged results can be obtained by the use of the tapered element when the beam is divided into six elements. It is to be noted that the maximum bending stress occurs at about 0.375 of the span from the free end.

The maximum bending stress would occur at the fixed end when the cantilever has a uniform constant thickness.

Finally, the usefulness of employing a consistent load vector is illustrated by the comparison of the results given in Figure (8). The tapered cantilever shown in this figure is subjected to a uniformly distributed load over the entire length. Figure 8 also gives the results for the vertical deflection at the free end when the cantilever is divided into 2, 4, 6 and 8 elements by using a static lumping process, a consistent load vector for a uniform beam element and the consistent load vector derived in the previous section for the tapered element. These results show the effectiveness of using an appropriate load vector; exact results are obtained regardless of number of elements used. In the case of the other two load idealisations, it appears that the results converge to a value differing by about 18%.

APPENDIX

Axially loaded tapered element

The transformation matrix [C] for this case is given by

$$[C] = \begin{bmatrix} 1 & \frac{1l}{a} \ln t_1 \\ 1 & \frac{1}{a} \ln t_2 \end{bmatrix}$$

and its inverse $[C]^{-1}$ is given by

$$[C]^{-1} \frac{1}{\ln t_2 - \ln t_1} \begin{bmatrix} \ln t_2 & -\ln t_1 \\ \frac{a}{l} & -\frac{a}{t} \end{bmatrix}$$

APPENDIX II

The transformation matrix [C] is

$$\begin{bmatrix} 1 & 0 & \frac{l_2}{2t_1(t_2-t_1)^2} & -\frac{l^3+2l^3 \ln t_1}{2(t_2-t_1)^3} \\ 1 & 0 & \frac{1}{2t_1^2(t_2-t_1} & \frac{l^2}{2t_1(t_2-t_1)^3} \\ 1 & & \frac{l^2}{2t_2(t_2-t_1)^3} & -\frac{l^3t_1+2l^3t_2\ln t_2}{2t_2(t_2-t_1)^3} \\ 0 & 1 & \frac{1}{2t_2^2(t_2-t_1)} & \frac{2l^2t_2-l^2t_1}{2t_2^2(t_2-t_1)^2} \end{bmatrix}$$

The terms of the (4 x 4) stiffness matrix [are given by

$$K(1,1) = \frac{a^3DEb}{\ldots 2A^2)}$$

$$K(1,2) = \frac{2l}{\ldots} K(1,1)$$

$$K(1,3) = -K(1,1)$$

$$K(1,4) = \frac{2l}{a}(A + \frac{aD}{2}) K(1,1)$$

$$K(2,2) = \frac{2l^2C}{a^2D} K(1,1)$$

$$K(2,3) = -K(1,2)$$

$$K(2,4) = -(1 + \frac{aA}{C}) K(2,2)$$

$$K(3,3) = K(1,1)$$

$$K(3,4) = -K(1,4)$$

$$K(4,4) = -\frac{azeb}{6l(2Q \quad zD)}$$

$$A = \frac{1}{t^2} - \frac{1}{2t^1} - \frac{l^1}{2t_2^2}$$

$$C = \ln \frac{t_2}{t_1} + \frac{2t_1}{t_2} - \frac{t_1^2}{2t_2^2}$$

$$D = \frac{1}{1} \quad \frac{1}{t_2^2}$$

$$Q = \frac{t_2}{2t_1^2} - \frac{1}{t_1} + \frac{1}{2t_2}$$

$$Z = \ln \frac{t_2}{t_1} - \frac{t_2^2}{t_1} + \frac{t_2^2}{2t_1^2} + \frac{3}{2}$$

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