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RESEARCH NOTE

# A Common Weight Multi-criteria Decision analysis-data Envelopment Analysis Approach with Assurance Region for Weight Derivation from Pairwise Comparison Matrices

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*ABSTRACT*

Deriving weights from a pairwise comparison matrix (PCM) is a subject for which a wide range of methods have been presented. This paper proposes a common weight multi criteria decision analysis-data envelopment analysis (MCDA-DEA) approach with assurance region for weight derivation from a PCM. The proposed method has a more discrimination power over the conventional methods for weight derivation from a pairwise comparison matrix. Furthermore, the proposed model has several merits over the competing approaches and lacks the drawbacks of the well-known Data Envelopment Analysis HP and data envelopment analysis / assurance region methods (DEA/AR) methods. Some numerical examples and a case study are taken from the literature in order to confirm the merits of the proposed method and its applications in multi criteria decision making. Results disclose the advantages of the proposed approach.

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## 1. INTRODUCTION

One of the most crucial steps in many decision making methods is the accurate estimation of the relevant data. This issue is particularly important in methods which need to elicit qualitative information from the decision makers (DMs). Very often qualitative data cannot be known in terms of absolute values. Therefore, many decision making methods attempt to determine the 'relative' importance, or weight, of the alternatives in terms of each criterion [1, 2].

The analytic hierarchy process (AHP) introduced by Saaty [3], is one of the most popular techniques in multi attribute decision making (MADM). This method has been widely applied in many fields of studies such as selection cargo terminals [4]. A number of approaches for weight derivation from a PCM have been suggested in the AHP literature with some drawbacks and advantages, but none of them can be declared as the best. Although the EVM is strongly recommended by Saaty [3, 5], there is no common agreement about its superiority. Most of the methods are based on some optimization approaches. In such approaches an objective function is firstly introduced. Then, the distance between an 'ideal' solution and the actual one is measured and the objective function is minimized [6]. Direct Least Squares Method (DLSM) minimizes the Euclidean Distance from the given comparison matrix under additive normalization constraints. The Weighted Least Squares Method (WLSM) is another optimization method which uses a modified Euclidean norm as an objective function [7]. Crawford [8] proposed the Logarithmic Least Squares Method (LLSM) which minimizes a logarithmic objective function. The Gradient Eigen Weight method (GEM) and a Least Distance Method (LDM) [9] and the Logarithmic Goal Programming Approach (LGPA) [10] are some other approaches in the literature of weight derivation from a PCM.

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In this paper, we propose a common weight MCDA-DEA model with assurance region for weight generation in the AHP applications. This method benefits from the basic advantages of the DEA-like models that is; 1) the accurate estimation of the qualitative information, 2) determining the relative importance of the alternatives in term of each criteria in a decision making problem and, 3) carrying out the weighting and aggregating steps simultaneously in an objective manner without the need for expert opinions. The proposed model is sensitive to changes in the elements of PCM. It employs all information of PCM which provides a reliable priority estimation to derive local weights as well. On the contrary to the existing DEAHP based models which use different set of coefficients in aggregation process to derive local weights, the proposed model uses a set of common coefficients. The common coefficients are very essential for fair comparison of entities in the crucial step in a decision making problem which is deriving appropriate weight from each PCM [11]. In addition, the proposed model significantly reduces the number of linear programming models, which should be solved to handle a typical application of AHP. In other words, to derive the local weights of a $n \times n$ pairwise comparison matrix, the proposed model requires solving just one linear programming, while other DEAHP based approaches need solving *n* linear programming models. The proposed method is one of the effective relevant ones for weight derivation from the view point of performance measurement. So, the model has several merits over the competing approaches and does not have the drawbacks of the well-known DEAHP and DEA/AR methods.

This paper is organized as follows. Section 2 presents a brief discussion on DEAHP method and its shortcomings via some numerical examples. The proposed approach and its prominent features are illustrated in Section 3. In Section 4, a number of numerical examples are provided and the results are compared with some competing methods including: DEAHP, DEA with assurance region (DEA/AR) and eigenvector method (EVM). Computational complexity of the proposed model is illustrated in Section 5, and some concluding remarks are discussed in Section 6.

## 2. DEAHP: THE CONCEPT AND SHORTCOMINGS

Suppose $A = (a_{ij})$ be a $n \times n$ pairwise comparison matrix of *n* factors (e.g., decision criteria or alternatives) with $a_{ii} = 1$ and $a_{ji} = 1/a_{ij}$ for $i \neq j$ and $W = (w_1, w_2, \ldots, w_n)^T$ denotes the corresponding weight vector where $w_i$ indicates the weight (relative importance) of *i*-th factor. DEAHP method views each factor in one row of a PCM, as a decision making unit, and the columns of the PCM as the outputs of these DMUs, and uses a dummy input that has a constant value of one for all DMUs to build an input oriented CCR model for each DMU [12, 13]. Therefore, each DMU has *n* outputs and one dummy constant input. The resulting efficiency score for each DMU is considered as the local weights of that DMU. The following linear programming model is used to estimate the local weighs of comparison matrix A [14]:

$$
\begin{aligned}
& Max\, w_k = \sum_{j=1}^{n} a_{kj} \upsilon_j^k \\
& s.t.\;\; u_1 = 1 \\
& \sum_{j=1}^{n} a_{ij} \upsilon_j^k - u_1 \leq 0 \qquad i = 1,2,\ldots,n \\
& u_1, v_j^k \geq 0, \qquad \forall k,\; j = 1,2,\ldots,n
\end{aligned}
\tag{1}
$$

where the optimal value of $w_k$ represents the local weight of $\mathrm{DMU}_k$ and $v_j^k$ denotes the associated weight with respect to $j^{\text{th}}$ column of matrix *A* when $\mathrm{DMU}_k$ is under evaluation. The superscript *k* is used to show that each DMU is evaluated by different vector of $V^k = (v_1^k, v_2^k, \ldots, v_n^k)$. Model (1) is iteratively solved for all DMUs to estimate the local weight vector $W = (w_1, w_2, \ldots, w_n)^T$ of matrix A. Then, two linear programming models are solved to aggregate local weights into final weights in a hierarchical structure. These models view the decision alternatives as DMUs and their local weights with respect to each criterion as outputs. The second model generates the global weights without considering any relation among the local weights of decision criteria, and is written as follows:

$$
\begin{aligned}
& Maximize\;\; w_k^g = \sum_{j=1}^{n} w_{kj}\, v_j^k \\
& s.t.\;\; u_1 = 1 \\
& \sum_{j=1}^{n} w_{ij}\, v_j^k - u_1 \leq 0 \qquad i = 1,2,\ldots,N \\
& u_1, v_j^k \geq 0, \qquad \forall k,\; j = 1,2,..N
\end{aligned}
\tag{2}
$$

where $w_k^g$ represents the global weight of $k^{\text{th}}$ alternative and $v_j^k$ denotes the associated weight of $j^{\text{th}}$ criterion when assessing $k^{\text{th}}$ DMU and $w_{ij}$ is the local weight of $i^{\text{th}}$ alternative (*i*=1, 2,…, *N*) with respect to $j^{\text{th}}$ criterion (*j*=1, 2,…,*n*). Model (2) is iteratively solved to determine the global weights of all alternatives. However, the relationship among the local weights of decision criteria, i.e., $d_j = v_j^k / v_1^k \; (j = 1,2,\ldots,n)$ can be imposed as additional constraints in model (2). It should be noted that the superscript $k$ indicates that these relations are used when $k^{th}$ DMU is under evaluation. Therefore, the following model is applied to determine the global weights of alternatives when criteria weights are important.

$$\begin{aligned} & Maximiz\ w_k^g = v_1^k \sum_{j=1}^{n} w_{kj} d_j \\ & s.t.\ u_1 = 1 \\ & v_1^k \left( \sum_{j=1}^{n} w_{ij} d_j \right) - u_1 \le 0 \quad i = 1,2,\ldots,N \\ & u_1, v_j^k \ge 0, \qquad \forall k,\ j = 1,2,\ldots,n \end{aligned} \tag{3}$$

DEAHP suffers from the following drawbacks:

1. Irrational and unrealistic local weights. For instance consider the following inconsistent matrix A whose consistency ratio is $CR = 0.3409 > 0.1$.

$$A = \begin{bmatrix} 1 & 4 & 3 & 1 & 3 & 4 \\ 1/4 & 1 & 7 & 3 & 1/5 & 1 \\ 1/3 & 1/7 & 1 & 1/5 & 1/5 & 1/6 \\ 1 & 1/3 & 5 & 1 & 1 & 1/3 \\ 1/3 & 5 & 5 & 1 & 1 & 3 \\ 1/4 & 1 & 6 & 3 & 1/3 & 1 \end{bmatrix}$$

   For such a highly inconsistent PCM, its priorities cannot be precisely estimated, but the ranking order of the six alternatives can be easily inferred. In fact, it can be easily understood from the pairwise comparison matrix A that $A_1$ is the most important alternative as all of its elements are greater than or equal to one. By eliminating $A_1$, in the reduced PCM, $A_5$ is obviously the most important alternative. So $A_5$ is ranked at the second place. Removing $A_5$ from further consideration, new reduced PCM is achieved, from which $A_2$ is slightly better than $A_6$. So, $A_2$ and $A_6$ are, respectively, the third and fourth most important alternatives. For the last two alternatives, as $a_{34} = 1/5$ and $a_{43} = 5$, hence $A_4$ is more important than $A_3$. So, the final ranking is $A_1 \succ A_5 \succ A_2 \succ A_6 \succ A_4 \succ A_3$. This ranking order is also validated by Saaty's eigenvector method [15], fuzzy programming method [6], correlation coefficient maximization approach (CCMA) and LP-GFW method [16]. But, the local weight generated by DEAHP is $W^T = (1,1,0.333,1,1,1)$ for this pairwise comparison matrix. In this manner, there is no discrimination among $A_1, A_2, A_4, A_5,$ and $A_6$.

2. Not using all information in an inconsistent PCM to derive local weights (i.e., information). For example, consider matrix *B*. As the results show in Table 1, model (1) uses only the information of one column of pairwise matrix B to calculate local weights of $B_2, B_3,$ and $B_4$.

$$B = \begin{bmatrix} 1 & 1 & 4 & 5 \\ 1 & 1 & 5 & 3 \\ 1/4 & 1/5 & 1 & 3 \\ 1/5 & 1/3 & 1/3 & 1 \end{bmatrix}$$

3. Being unaffectedor insensitive to some comparisons. For instance, consider matrices C and D. It is clear that these two matrices are different in $c_{12}$ and $d_{12}$. But the DEAHP results in a same weight vector (1, 0.6, 0.2) for two matrices C and D. It is worth mentioning that the weight vector (1, 0.6, 0.2) is the normalization of the last column of the inconsistent matrices C and D. This fact means that the DEAHP uses only some parts of the information of a PCM. This drawback restricts the application of the DEAHP.
4. Also, overestimation of some local weights is another disadvantage of the DEAHP method [17].

Because of the drawbacks mentioned above, some researchers proposed new methods like DEA/AR model [17]. This method considers an assurance region for the variables in the DEAHP model and as a result uses all the information in the pairwise comparison matrix. In the next section, the proposed common weight MCDA-DEA method for weight derivation is elaborated. It is able to overcome the drawbacks of the DEAHP method and at the same time has some advantages over DEA/AR model. To demonstrate the superiority of the proposed method, it is compared to the DEA/AR model through an indicator called fitting performance (FP). The results show that the proposed model performs better than the traditional DEAHP and the DEA/AR methods.

$$C = \begin{bmatrix} 1 & 2 & 5 \\ 1/2 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 9 & 5 \\ 1/9 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix}$$

**TABLE 1.** Local weights of matrix B

| | $\upsilon_1$ | $\upsilon_2$ | $\upsilon_3$ | $\upsilon_4$ | **Local weight** |
|---|---|---|---|---|---|
| $w_1$ | 0 | 0 | 0.154 | 0.078 | 1 |
| $w_2$ | 0 | 0 | 0.2 | 0 | 1 |
| $w_3$ | 0 | 0 | 0 | 0.2 | 0.6 |
| $w_4$ | 0 | 1 | 0 | 0 | 0.333 |

## 3. PROPOSED MODEL

This paper proposes a common weight multi criteria decision analysis-data envelopment analysis (MCDA-DEA) approach for weight derivation from a PCM used in the analytic hierarchy process (AHP). The merits of the proposed model are as follows. First, it uses a set of common weights which leads to more rational comparisons between entities as it applies the same weight for all variables. Second, considering an assurance region for PCM's elements increases its discriminating power and provides better priority estimation than DEAHP method and the current models in the literature. Third, it needs less computational efforts in comparison with the DEA-based models. Besides, the proposed approach overcomes the drawbacks of the well-known DEAHP method.

### 3. 1. Deriving the Local Weights of a PCM

In contrast with the models mentioned earlier, the proposed model uses a common weight vector $(\upsilon_j)$ for all DMUs in order to obtain the weight vector from a PCM. In the traditional DEAHP method, all local weights are provided by applying different sets of weight vectors from a pairwise comparison matrix. Instead, in the proposed approach, all local weights are derived from aggregating a set of common weights which enables a fair comparison among them. The classical DEA models such as model (1) have the flexibility to provide the coefficients of decision variables, i.e., $\upsilon_j^k$, in its own favor for maximizing its own local weights. This flexibility may identify a DMU to be efficient with local weight of one by giving an extremely high coefficient to a criterion which has extremely good performance and an extremely small coefficient to that which has extremely bad performance.

However, such extreme values for coefficients are unrealistic and cause model (1) to have a poor discriminating power. However, our proposed model seeks for the common coefficients of all DMUs simultaneously by preventing each particular DMU to choose the coefficients in its own favor. In this paper, we propose two different models to derive the weight vectors. Model (9) is used to derive the local weights and model (10) is applied to aggregate the local weights into global weights in order to get the final ranking of the alternatives. Hatefi and Torabi [18] proposed a common weight MCDA-DEA model to create composite indicators (CIs) which had the power to discriminate those entities (DMUs) receiving CI score of 1 (Despite the other DEA-like models). Suppose there are $m$ DMUs and $n$ criteria. The common weight MCDA-DEA model to create CIs [18] is as follows:

$$\begin{aligned} & Min\, M \\ & S.t.\ M - d_i \ge 0;\ i = 1,2,\ldots,m \\ & \quad \sum_{j=1}^{n} w_j a_{ij} + d_i = 1;\ i = 1,2,\ldots,m \\ & \quad w_j \ge \varepsilon,\ j = 1,2,\ldots,n;\ d_i \ge 0;\ i = 1,2,\ldots,m \end{aligned} \tag{4}$$

where $w_j$ denotes the common weight of criterion "$j$" among all entities and $a_{ij}$ is the performance of $DMU_i$ with respect to criterion $j$. Also, $d_i$ denotes the efficiency deviation of $i^{th}$ DMU and $M$ is the maximum efficiency deviation among non-efficient DMUs. Using model (4), the CI (total performance) of $i^{th}$ entity is calculated by $CI_i = 1 - d_i;\ i = 1,2,\ldots,m$. According to model (4), model (5) can be reformulated for using in a $n \times n$ pairwise comparison matrix as follows:

$$\begin{aligned} & Min\, M \\ & S.t.\ M - d_i \ge 0;\ i = 1,2,\ldots,n \\ & \quad \sum_{j=1}^{n} \upsilon_j a_{ij} + d_i = 1;\ i = 1,2,\ldots,n \\ & \quad \upsilon_j, d_i \ge 0,\ i,j = 1,2,\ldots,n \end{aligned} \tag{5}$$

Note that as PCM is a $n \times n$ matrix, so "$i$" and "$j$" have the same dimension, and $d_i$ can be converted to $d_j$ in model (5). Using model (5), the weight of $i^{th}$ DMU is $w_i = 1 - d_i;\ i = 1,2,\ldots,m$. The difference between models (4) and (5) is the assurance region for $\upsilon_j$. So, we first develop an assurance region for this variable. According to [17] the assurance region for $\upsilon_j$ is:

$$\frac{w_j}{\beta} \le v_j \le \frac{w_j}{n};\ j = 1,2,\ldots,n \tag{6}$$

where:

$$\beta = \min\left\{ \max_i \left( \frac{1}{r_i \sum_{j=1}^{n} a_{ij} r_j} \right), \max_i \left( \frac{1}{c_j \sum_{i=1}^{n} a_{ij} c_i} \right) \right\} \tag{7}$$

And $r_1,\ldots,r_n$ and $c_1,\ldots,c_n$ are the summations of rows and columns of the PCM $\left(A = \lfloor a_{ij} \rfloor_{n\times n}\right)$, respectively. So:

$$\frac{1-d_i}{\beta} \le v_j \le \frac{1-d_i}{n} \tag{8}$$

By combining model (5) and formula (8), the proposed model is reformulated as:

$$\begin{aligned} & Min\, M \\ & S.t.\ M - d_i \ge 0;\ i = 1,2,\ldots,n \\ & \quad \sum_{j=1}^{n} a_{ij}\upsilon_j + d_i = 1;\ i = 1,2,\ldots,n \\ & \quad n\upsilon_j \le 1 - d_j;\ j = 1,2,\ldots,n \\ & \quad 1 - d_j \le \beta\upsilon_j;\ j = 1,2,\ldots,n \\ & \quad \upsilon_j, d_i \ge 0,\ i,j = 1,2,\ldots,n \end{aligned} \tag{9}$$

Model (9) confirms that common weight MCDA-DEA model produces true weights for perfectly consistent PCMs. As it was mentioned earlier, by using model (9), the local weight of $i^{th}$ DMU is derived by $w_i = 1 - d_i^*; i = 1,2,\ldots,m$. For inconsistent PCMs, according to the two equations for $\upsilon_j$, rational, logical and intuitive weights which are accordant with decision maker's subjective judgment will be produced. This fact will be explained through numerical examples in the next section.

**3. 2. Deriving the Final Weights of a PCM** In this section, a model is proposed to get the final ranks of the alternatives. Suppose $L_{ij}$ represents the local weight of the alternative *i* with respect to criterion *j*. In contrast with the traditional AHP which applies the weighted sum method to get final weights, DEA method considers the DEA matrix of local weights. In the DEA local weights aggregation process, two different cases can be applied; (1): without considering the local weights of criteria and (2): considering the local weights of criteria. (1): in this case a simple DEA model is used to get the final weights. Since the importance of criteria are generated automatically in this process, the local weights of criteria are not necessary in this condition.
(2): The importance measure of criteria is applied in DEA methods using the assurance region by additional constraints determine relationships among multipliers in the original DEA model.

In this regard, the importance of criteria is embodied in the form of multipliers $\upsilon_1 = K_j \upsilon_j; j = 2,\ldots..,m$, where $v_1$ is the variable obtained by model (9) in the process of deriving local weights with respect to the goal in an AHP problem. For example, if criterion 1 is half and quadruple as important as criteria 2 and 3, respectively, then we have $K_2 = 1/2$ and $K_3 = 4$. Theorem 1 proves this fact. In this case, it can be proved that the final weights of alternatives estimated by MCDA-DEA method are proportional to the weighted sum of local weights as shown in Theorem 1.

**Theorem 1.** When the importance of criteria is incorporated in a DEA-based model by using additional constraints mentioned in case (2), the final weights of alternatives is proportional to the weighted sum $\sum_{j=1}^{n} K'_j L_{ij}$ for alternative *i*.

**Proof.** Suppose matrix $[L_{ij}]$, in which $L_{ij}$ represents the local weight of alternative *i* with respect to criterion *j* $(i = 1,2,\ldots.,m, j = 1,2,\ldots,n)$. The following linear programming model is used to derive final weights of alternatives.

$$\begin{aligned} & Min\, M \\ & s.t. \\ & M - d_i \geq 0;\; i = 1,2,\ldots,n \\ & \sum_{j=1}^{n} \upsilon_j L_{ij} + d_i = 1;\; i = 1,2,\ldots,n \\ & \upsilon_j \geq 0;\; j = 1,2,\ldots,n \end{aligned} \tag{10}$$

After incorporating the importance of criteria by adding $\upsilon_1 = K_j \upsilon_j$ or $\upsilon_j = K'_j \upsilon_1;\; j = 1,2,\ldots.,n$ constraints, model (10) can be reformulated as follows:

$$\begin{aligned} & Min\, M \quad s.t. \\ & M - d_i \geq 0;\; i = 1,2,\ldots,m \\ & \upsilon_1 \sum_{j=1}^{n} K'_j L_{ij} + d_i = 1;\; i = 1,2,\ldots,m \quad \upsilon_1 \geq 0 \end{aligned} \tag{11}$$

and the final weights are obtained by $w_i^f = 1 - d_i^*$ after solving model (10). At optima, the constraint $\upsilon_1^* \sum_{j=1}^{n} K'_j L_{ij} = 1 - d_i^*;\; i = 1,2,\ldots,m$ should be held. This means that the final weight of the $i^{th}$ alternative, that is $1 - d_i^*$, is proportional to the weighted sum of local weights, i.e., $\sum_{j=1}^{n} K'_j L_{ij}$ for alternative *i*.

## 4. NUMERICAL EXPERIMENTS

In this section, some numerical examples are provided to compare the proposed model with different approaches which are Eigenvector method (EVM), DEAHP and DEA/AR. Furthermore, the following discussion shows the advantages of the proposed method over the disadvantages of the DEAHP method. The fitting performance measure is a witness for the concession of the common weight MCDA-DEA model. Furthermore, the proposed model outperforms the DEA/AR and EM models with respect to FP measure. Finally, the assurance regions of the numerical examples are presented for more details.

**4. 1. A Comparison between EVM, DEAHP, DEA/AR AND the Proposed Model** Consider these four matrices, A, B, C, D in which the elements are the importance of the alternatives in comparison to each other. The values of these matrices are expert based. Table 2 shows the local weights produced by four different methods including EM, DEAHP, DEA/AR and the proposed model. The difference between matrices C and D is the elements $c_{12}$ and $d_{12}$. For the pairwise comparison matrices C and D, the local weights produced by the common weight MCDA-DEA model are sensitive enough to the changes of elements $c_{12}$ and $d_{12}$. However, the DEAHP method provides the similar local weights for matrices C and D, and is unable to distinguish these changes to derive local weights. Thus, it is clear that the proposed model reacts well to the changes in pairwise comparisons.

As Table 3 shows, it is obvious that DEAHP uses only part of the PCM's information for weight derivation. This is the reason why DEAHP model concludes in too many decision alternatives as efficient DMUs. According to the constraints in the proposed model for $\upsilon_j$, it applies all the information (values) in a PCM for weight derivation.

$$A=\begin{bmatrix} 1 & 4 & 3 & 1 & 3 & 4 \\ 1/4 & 1 & 7 & 3 & 1/5 & 1 \\ 1/3 & 1/7 & 1 & 1/5 & 1/5 & 1/6 \\ 1 & 1/3 & 5 & 1 & 1 & 1/3 \\ 1/3 & 5 & 5 & 1 & 1 & 3 \\ 1/4 & 1 & 6 & 3 & 1/3 & 1 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & 1 & 4 & 5 \\ 1 & 1 & 5 & 3 \\ 1/4 & 1/5 & 1 & 3 \\ 1/5 & 1/3 & 1/3 & 1 \end{bmatrix}$$

$$C=\begin{bmatrix} 1 & 2 & 5 \\ 1/2 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix} \qquad D=\begin{bmatrix} 1 & 9 & 5 \\ 1/9 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix}$$

In order to check the information used by the two models, Table 3 shows the value of the decision variables in different pairwise comparison matrices. According to Table 3, it can be seen clearly that the DEAHP applies only some part of a PCM's information for weight derivation as nearly all the values for $\upsilon_j$ are equal to zero. So the results of this method are not reliable and conclude in too many decision alternatives as efficient DMU.

Likewise, the fitting performance indicator shown in Table 4 is applied to check the quality of the local weights derived by the proposed model. The fitting performance (FP) is measured by the following Euclidean distance [17]:

$$FP=\sqrt{1/n^2\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}-w_i/w_j)^2} \tag{12}$$

As it is shown in Table 4 above, the fitting performance indicator for the proposed model is less than those of DEA/AR model. This indicator is a witness for the concession of the common weight MCDA-DEA model. Furthermore, our proposed model outperforms the DEA/AR and EM models with respect to FP measure. It is observed that DEAHP performs well for pairwise matrix C which is nearly perfectly consistent, but it works poorly for the other inconsistent pairwise comparison matrices. Table 5 shows some details in the way of weight derivation from PCM for the matrices A, B, C and D. The first column indicates the value of parameter $\beta$ according to formula (7),so that the lower and upper bound of the parameter $\upsilon_j$ is obtained in the second and third columns, respectively according to formula (8). It is obvious that according to the two equations for $\upsilon_j$, rational, logical and intuitive weights which are accordant with decision maker's subjective judgment is produced.

**4. 2. An Illustrative Example** In this section, an example is provided to illustrate the potential applications of the proposed method and shows its merits over other priority methods. From the example, it is inferred that the proposed model uses more information of a PCM than DEAHP. The next advantage is that the proposed method is able to produce more logical and reasonable ranking of the alternatives even for highly inconsistent PCMs in comparison to DEAHP and other priority approaches. Figure 1 shows the hierarchical structure of the problem investigated by Ramanthan [14] using DEAHP. In addition, the comparison matrices for four criteria and three alternatives and their relevant weight vectors and the corresponding results are presented in Tables 6 and 7. To obtain the final weights of the alternatives, we consider two cases. In the first case, the importance measures of criteria are used to calculate the final weights. Using the local weights of criteria reported in Table 5, three additional constraints, i.e., $\upsilon_1=1/0.9585\upsilon_2$, $\upsilon_1=1/0.2146\upsilon_3$ and $\upsilon_1=1/0.2049\upsilon_4$ are used in model (10) to get the final weights of the alternatives (See the third column of Table 7).

In the second case, the final weights driven from the weighted sum method are also calculated. For instance, for alternative $A_1$, the weighted sum is calculated as follows:
[(0.4329×1)+(0.0862×0.9585)+(1×0.2146)+(1×0.2049)] =0.935.

In a similar way, the weighted sums for alternatives $A_2$ and $A_3$ are calculated as 2.140 and 0.490, respectively. The normalized values of the weighted sum results are reported in the second column of Table 6. The last column of Table 7 reports the final weights driven from the conventional DEAHP method. All results of Table 7 affirms that the ranking vector is: $A_2 \succ A_1 \succ A_3$.

## 5. COMPUTATIANAL COMPLEXITY

One of the most important advantages of the proposed common weight MCDA-DEA method with assurance region is the reduction in the required computational efforts. For a pairwise comparison matrix having '*n*' alternatives, the DEA-based methods require to solve '*n*' linear programming to derive the local weights of the alternatives. Then, when it is compared to EVM, the DEA-based methods apply more computational attempts. However, the method proposed in this paper requires one linear programming for weight derivation which leads to fewer computations. Suppose an AHP problem with $m$ alternatives and $n$ criteria. Our proposed model requires to solve $n$+2 linear programming models to derive final ranking of alternatives, while the existing DEAHP based models in the literature, i.e., the conventional DEAHP [14] and the DEA models developed in [17, 19], solve $m+n+mn$ linear programming models to obtain final results. Therefore, the proposed method significantly diminishes the number of required linear programming models to be solved. In the practical application of the AHP method, when the size of pairwise comparison matrixes increases, the consistency ratio decreases due to the large number of pairwise comparisons that experts should be performed. Therefore, utilization of AHP method is not recommended for solving the large size problems. Consequently, we do not recommend using the proposed method for weight derivations in the large size problems.

**TABLE 2.** Local weights of the PCMs (A, B, C, D) inferred from four different models

| Matrix | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ |
|---|---|---|---|---|---|---|
| | | | EM model | | | |
| A | 0.320 | 0.139 | 0.034 | 0.128 | 0.237 | 0.139 |
| B | 0.400 | 0.393 | 0.127 | 0.078 | - | - |
| C | 0.582 | 0.309 | 0.109 | - | - | - |
| D | 0.764 | 0.149 | 0.087 | - | - | - |
| | | | DEAHP model | | | |
| A | 1.000 | 1.000 | 0.333 | 1.000 | 1.000 | 1.000 |
| B | 1.000 | 1.000 | 0.600 | 0.333 | - | - |
| C | 1.000 | 0.600 | 0.200 | - | - | - |
| | | 0.600 | 0.200 | - | - | - |
| D | 1.000 | | | | | |
| | | | DEA/AR model | | | |
| A | 1.000 | 0.542 | 0.124 | 0.439 | 0.818 | 0.532 |
| B | 1.000 | 0.996 | 0.325 | 0.201 | - | - |
| C | 1.000 | 0.531 | 0.188 | - | - | - |
| | | 0.213 | 0.125 | - | - | - |
| D | 1.000 | | | | | |
| | | | Proposed model | | | |
| A | 1.000 | 0.536 | 0.124 | 0.463 | 0.768 | 0.526 |
| B | 1.000 | 0.968 | 0.323 | 0.201 | - | - |
| C | 1.000 | 0.531 | 0.188 | - | - | - |
| D | 1.000 | 0.207 | 0.123 | - | - | - |

**TABLE 3.** The values of decision variables $v_j$ from different models

| Comparison matrix | Objective function | DEAHP | | | | | | Proposed model | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\upsilon_1$ | $\upsilon_2$ | $\upsilon_3$ | $\upsilon_4$ | $\upsilon_5$ | $\upsilon_6$ | $\upsilon_1$ | $\upsilon_2$ | $\upsilon_3$ | $\upsilon_4$ | $\upsilon_5$ | $\upsilon_6$ |
| A | $w_1$ | 0 | 0.182 | 0 | 0 | 0.091 | 0 | | | | | | |
| | $w_2$ | 0 | 0 | .143 | 0 | 0 | 0 | | | | | | |
| | $w_3$ | 1.000 | 0 | 0 | 0 | 0 | 0 | 0.167 | 0.051 | 0.021 | .080 | .095 | .050 |
| | $w_4$ | .348 | 0 | .130 | 0 | 0 | 0 | | | | | | |
| | $w_5$ | 0 | .200 | 0 | 0 | 0 | 0 | | | | | | |
| | $w_6$ | 0 | 0 | .029 | .250 | .221 | 0 | | | | | | |
| B | $w_1$ | 0 | 0 | .154 | .078 | - | - | | | | | | |
| | $w_2$ | 0 | 0 | .200 | 0 | - | - | 0.250 | 0.242 | 0.066 | 0.049 | - | - |
| | $w_3$ | 0 | 0 | 0 | .200 | - | - | | | | | | |
| | $w_4$ | 0 | 1.000 | 0 | 0 | - | - | | | | | | |
| C | $w_1$ | 0 | 0 | .200 | - | - | - | | | | | | |
| | $w_2$ | 0 | 0 | .200 | - | - | - | 0.333 | 0.177 | 0.063 | - | - | - |
| | $w_3$ | 0 | 0 | .200 | - | - | - | | | | | | |
| D | $w_1$ | 0 | .111 | .200 | - | - | - | | | | | | |
| | $w_2$ | 0 | 0 | .200 | - | - | - | .333 | .052 | .039 | - | - | - |
| | $w_3$ | 1.000 | 0 | 0 | - | - | - | | | | | | |

**TABLE 4**. Fitting performance by different local weights

| | CR | FP | | | |
|---|---|---|---|---|---|
| | | EM | DEAHP | DEA/AR | Proposed model |
| A | 0.229 | 1.579 | 1.551 | 1.588 | 1.432 |
| B | 0.088 | 0.811 | 1.182 | 1.792 | 0.797 |
| C | 0.003 | 0.128 | 0.116 | 0.128 | 0.127 |
| D | 0.280 | 1.854 | 2.449 | 1.804 | 1.792 |

**TABLE 5**. The values related to the assurance region

| Comparison matrix | $\beta$ | Lower bound | Upper bound |
|---|---|---|---|
| A | 9.3944 | 0.1064 | 0.1667 |
| B | 4.8850 | 0.2047 | 0.2500 |
| C | 3.0556 | 0.3273 | 0.3333 |
| D | 3.9780 | 0.2514 | 0.3333 |

**TABLE 6.** Pairwise comparison matrices for criteria and alternatives and their proposed weight vectors

(A)Comparisons of criteria with respect to goal

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | Local weights |
|---|---|---|---|---|---|
| $C_1$ | 1 | 1 | 4 | 5 | 1 |
| $C_2$ | 1 | 1 | 5 | 3 | 0.9585 |
| $C_3$ | 1/4 | 1/5 | 1 | 3 | 0.2146 |
| $C_4$ | 1/5 | 1/3 | 1/3 | 1 | 0.2049 |

Consistency ratio $CR = 0.088$

(B) Comparisons of alternatives with respect to $C_1$

| | $A_1$ | $A_2$ | $A_3$ | Local weights |
|---|---|---|---|---|
| $A_1$ | 1 | 1/3 | 5 | 0.4329 |
| $A_2$ | 3 | 1 | 7 | 1.0000 |
| $A_3$ | 1/5 | 1/7 | 1 | 0.1120 |

Consistency ratio $CR = 0.055$

(C) Comparisons of alternatives with respect to $C_2$

| | $A_1$ | $A_2$ | $A_3$ | Local weights |
|---|---|---|---|---|
| $A_1$ | 1 | 1/9 | 1/5 | 0.0862 |
| $A_2$ | 9 | 1 | 4 | 1.0000 |
| $A_3$ | 5 | 1/4 | 1 | 0.3290 |

Consistency ratio $CR = 0.061$

(D) Comparisons of alternatives with respect to $C_3$

| | $A_1$ | $A_2$ | $A_3$ | Local weights |
|---|---|---|---|---|
| $A_1$ | 1 | 2 | 5 | 1.0000 |
| $A_2$ | 1/2 | 1 | 3 | 0.5314 |
| $A_3$ | 1/5 | 1/3 | 1 | 0.1882 |

Consistency ratio $CR = 0.003$

(E) Comparisons of alternatives with respect to $C_4$

| | $A_1$ | $A_2$ | $A_3$ | Local weights |
|---|---|---|---|---|
| $A_1$ | 1 | 3 | 9 | 1.0000 |
| $A_2$ | 1/3 | 1 | 3 | 0.3333 |
| $A_3$ | 1/9 | 1/3 | 1 | 0.1111 |

Consistency ratio $CR = 0$

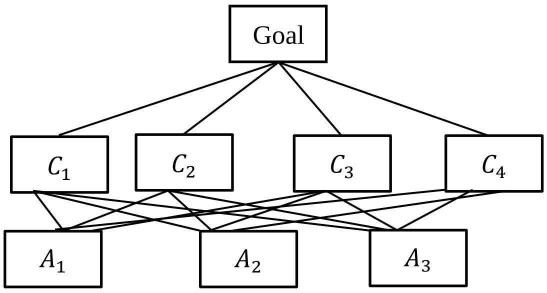


**Figure 1.**The AHP model

**TABLE 7**. Final weights provided by the weighted sum and the proposed method

| | **Weighted sum method** | **Proposed method** | ***DEAHP weights*** |
|---|---|---|---|
| $A_1$ | 0.4368 | 0.7032 | 0.7120 |
| $A_2$ | 1.0000 | 1.0000 | 1.0000 |
| $A_3$ | 0.2291 | 0.1543 | 0.3462 |

**TABLE 8**. A qualitative comparison of different models

| No. of LP to be solved | | Type of weights |
|---|---|---|
| EVM | -- | Changeable |
| DEAHP | $m+n+mn$ | Changeable |
| DEA/AR | $m+n+mn$ | Changeable |
| Proposed model | $n+2$ | Common |

Table 8 provides qualitative comparison of the proposed model with the existing approaches for weight derivation from a PCM. It shows several merits of the proposed method when comparing it with the methods mentioned in the literature.

## 6. CONCLUSION

In this paper, we propose a common weight MCDA-DEA method with assurance region for weight derivation from a pairwise comparison matrix with a more discriminating power over the existing ones (AHP, DEAHP, DEA/AR). In contrast to the competing models, the proposed model uses a common weight vector $\upsilon_j$ for all alternatives in order to obtain the final ranking in an AHP problem.

The merits of the proposed model when compared to the DEAHP [14] and DEAHP/AR [17] are summarized as follows:

- The proposed method has more distinguishing power than DEAHP method. The alternatives are evaluated through the common $\upsilon_j$ to derive the local and/ or final weights. In other words, the proposed method prevent each particular DMU to choose the coefficients $\upsilon_j$ in its own favor (in contrast with the DEAHP) to derive the local and/ or final weights
- As the common vector of coefficients is used to derive the local and/or final weights, the evaluation of alternatives is fairer, while, DMUs are actually evaluated by different coefficient vectors in the DEAHP and DEAHP/AR methods.
- All the information in the pairwise comparison matrix will be used in our proposed method to derive local weights and information loss will be eliminated due to employing assurance region for the variable $\upsilon_j$.
- The proposed method significantly reduces the number of linear programing which should be solved to handle an AHP problem, when it is compared to DEAHP based methods introduced in the literature such as DEAHP [14] and DEAHP/AR [17] and [19].

The proposed model is applied for four different pairwise comparison matrices. The power of the model is tested by fitting performance (FP) parameter over DEA/AR and EVM methods. It is illustrated that this method uses all the information of the pairwise comparison matrix in contrast with the DEAHP model. In addition, this method determines one of the alternatives (DMUs) as the best entity unlike the DEAHP model (which irrationally determines most of the DMUs as the efficient ones). Finally, this method is sensitive enough to the changes of the pairwise comparison matrix elements, which is not the case for DEAHP method.

## 7. REFERENCES


1. Tavakkoli-Moghaddam, R., Heydar, M. and Mousavi, S., "An integrated ahp-vikor methodology for plant location selection", ***International Journal of Engineering-Transactions B: Applications***, Vol. 24, No. 2, (2011), 127-137.
2. Mousavi, S., Makoui, A., Raissi, S. and Mojtahedi, S., "A multi-criteria decision-making approach with interval numbers for evaluating project risk responses", ***International Journal of Engineering-Transactions B: Applications***, Vol. 25, No. 2, (2012), 121-130.
3. Saaty, T.L., "The analytic hierarchy process: Planning, priority setting, resources allocation", ***New York: McGraw***, (1980).
4. Mirzaei, E., Minatour, Y., Bonakdari, H. and Javadi, A., "Application of interval-valued fuzzy analytic hierarchy process approach in selection cargo terminals, a case study", ***International Journal of Engineering-Transactions C: Aspects***, Vol. 28, No. 3, (2014), 387-395.
5. Saaty, T.L., "Decision-making with the ahp: Why is the principal eigenvector necessary", ***European Journal of Operational Research***, Vol. 145, No. 1, (2003), 85-91.
6. Mikhailov, L., "A fuzzy programming method for deriving priorities in the analytic hierarchy process", ***Journal of the Operational Research Society***, (2000), 341-349.
7. Chu, A., Kalaba, R. and Spingarn, K., "A comparison of two methods for determining the weights of belonging to fuzzy sets", ***Journal of Optimization Theory and Applications***, Vol. 27, No. 4, (1979), 531-538.
8. Crawford, G., "The geometric mean procedure for estimating the scale of a judgement matrix", ***Mathematical Modelling***, Vol. 9, No. 3, (1987), 327-334.
9. Cogger, K. and Yu, P., "Eigenweight vectors and least-distance approximation for revealed preference in pairwise weight ratios", ***Journal of Optimization Theory and Applications***, Vol. 46, No. 4, (1985), 483-491.
10. Joseph, A., "Generating consensus priority point vectors: A logarithmic goal programming approach", ***Computers & Operations Research***, Vol. 26, No. 6, (1999), 637-643.
11. Torabi, S. and Shokr, I., "A common weight data envelopment analysis approach for material selection", ***International Journal of Engineering-Transactions C: Aspects***, Vol. 28, No. 6, (2015), 913-922.
12. Charnes, A., Cooper, W.W. and Rhodes, E., "Measuring the efficiency of decision making units", ***European Journal of Operational Research***, Vol. 2, No. 6, (1978), 429-444.
13. Cooper, W.W., Seiford, L.M. and Tone, K., "Data envelopment analysis: A comprehensive text with models, applications, references and dea-solver software, Springer Science & Business Media, (2007).
14. Ramanathan, R., "Data envelopment analysis for weight derivation and aggregation in the analytic hierarchy process", ***Computers & Operations Research***, Vol. 33, No. 5, (2006), 1289-1307.
15. Saaty, T.L., "Fundamentals of decision making and priority theory with the analytic hierarchy process, Rws Publications, Vol. 6, (2000).
16. Wang, Y.-M., Parkan, C. and Luo, Y., "Priority estimation in the ahp through maximization of correlation coefficient", ***Applied Mathematical Modelling***, Vol. 31, No. 12, (2007), 2711-2718.
17. Wang, Y.-M., Chin, K.-S. and Poon, G.K.K., "A data envelopment analysis method with assurance region for weight generation in the analytic hierarchy process", ***Decision Support Systems***, Vol. 45, No. 4, (2008), 913-921.
18. Hatefi, S. and Torabi, S., "A common weight mcda–dea approach to construct composite indicators", ***Ecological Economics***, Vol. 70, No. 1, (2010), 114-120.
19. Wang, Y.-M., Parkan, C. and Luo, Y., "A linear programming method for generating the most favorable weights from a pairwise comparison matrix", ***Computers & Operations Research***, Vol. 35, No. 12, (2008), 3918-3930.

# A Common Weight Multi-criteria Decision analysis-data Envelopment Analysis Approach with Assurance Region for Weight Derivation from Pairwise Comparison Matrices

**RESEARCH NOTE**

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**چکیده**

استخراج اوزان از یک ماتریس مقایسات زوجی موضوعی است که طیف وسیعی از روشها برای آن ارایه شده است. در این مقاله یک رویکرد تحلیل تصمیم گیری چند معیاره وزن مشترک–تحلیل پوششی داده‌ها با ناحیه اطمینان برای استخراج اوزان از یک ماتریس مقایسات زوجی پیشنهاد شده است. روش پیشنهادی قدرت تمایز بیشتری نسبت به روشهای مرسوم برای استخراج اوزان دارد. علاوه بر این روش، پیشنهادی مزایای متعددی در مقایسه با روش‌های رقیب دارد و معایب روش‌های DEAHP و DEA/AR را برطرف می‌کند. به منظور تایید مزایای روش پیشنهادی و کاربرد آن در تصمیم گیری چندمعیاره چند مثال عددی و یک مطالعه موردی از ادبیات موضوع گرفته شده است. نتایج مزایای روش پیشنهادی را آشکار می‌کنند.

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