PULSATILE MOTION OF BLOOD IN A CIRCULAR TUBE OF VARYING CROSS-SECTION WITH SLIP FLOW

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Abstract Pulsatile motion of blood in a circular tube of varying cross-section has been developed by considering slip flow at the tube wall and the blood to be a non- Newtonian biviscous incompressible fluid. The tube wall is supposed to be permeable and the fluid exchange across the wall is accounted for by prescribing the normal velocity of the fluid at the tube wall. The tangential velocity of the fluid at the tube wall is also accounted in the present investigation. A perturbation technique has been carried out for low Reynolds number flow and for small amplitude of oscillation. The effects of slip parameter, leakage parameter, Reynolds number and apparent viscosity coefficient on the streamlines, wall shear stress and pressure drop have been discussed and shown graphically for suction and injection respectively.

Keywords Pulsatile motion; slip flow; biviscous fluid; leakage parameter; Womersle`s parameter.

چکیده حرکت ضربانی خون در لوله دایراه ای شکل با سطوح مقطع متفاوت با در نظر گرفتن شیب جریان در دیواره لوله و با فرض خون به عنوان یک سیال تراکم ناپذیر ویسکوز غیر نیوتنی مورد بررسی قرار گرفته است. دیواره لوله نفوذ پذیر فرض شده است و حرکت سیال در مقطع عرضی دیواره به وسیله تعیین سرعت نرمال سیال در دیواره لوله محاسبه میگردد. همچنین در تحقیق حاضر سرعت مماسی سیال در دیواره لوله محاسبه میگردد. روش اختلال برای سیالات با اعداد رینولدز پایین ودامنه نوسانات کم بکار گرفته شده است. تاثیر پارامتر شتاب، نشتی، عدد رینولدز و ضریب وسکوزیته مشخص بر روی خطوط جریان، تنش برشی دیواره و افت فشار مورد بحث قرار گرفته و به ترتیب برای مکش و تزریق رسم گردیده اند.

1. INTRODUCTION

The study of pulsatile flow over boundaries with deformation has been attracted by researchers because of its importance in understanding the fluid mechanical aspects of blood flow. The pulsatility of blood flow is one of the most important factors in Biofluid mechanics. The rhythmic action of the heart causes this pulsatile nature in blood, which is influenced by some properties of blood and blood vessels. Womersley [1, 2] considered the oscillatory flow in a cylindrical tube with uniform cross-section. Lee and Fung [3] studied the flow of blood through an artery with an axisymmetric stenosis taking blood as a Newtonian fluid. Bitoun and Bellet [6] studied pulsatile flow of blood with reference to stenosis in microcirculation. Pulsatile flow through circular tubes with varying cross-section has been investigated by Rao and Devanathan [4] and also by Schneck and Ostrach [5]. In these studies the tube wall is taken to be impermeable. However, in the case of small blood vessels, the permeability of the walls becomes important. Low Reynolds number flow in slowly varying axisymmetric tubes has been analysed by Manton [7]. Radhakrishnamacharya et al. [8] and Prasad et al. [9] studied the pulsatile flow of blood in circular tubes of varying cross-section with suction/injection. But the non- Newtonian property is not taken into consideration in these studies. As blood shows the remarkable non- Newtonian property in low shear rate and the shear rate is low in the downstream side of the stenosis, it is considered that the analysis of the flow pattern near stenosis should include the non- Newtonian property of blood. It is a mixture of plasma and blood cells and this suspension of blood has recently become the object of scientific research of Chow [10], Hill and Bedford [11], Srivastava and Agarwal [12]. Nakayama and Sawada [13] studied the flow of a non- Newtonian fluid through an axisymmetric stenosis numerically. The pulsatile flow of a non- Newtonian biviscous fluid through a tube with varying cross-section and non-permeable walls in presence of external magnetic field has been analysed by Elnaby et al. [14]. Sanyal et al. [15] investigated the pulsatile flow of biviscous fluid through a tube of varying cross-section with suction/injection. But they considered no effect of slip velocity at the wall of the tube and so the effect of slip velocity has been neglected. Raoufpanah et al. [16] studied the effect of slip condition on the characteristic of flow in ice melting process. Recently, Das [17] discussed the heat transfer peristaltic transport with slip condition in an asymmetric porous channel.

Here, our main object is to study the pulsatile motion of blood in a circular tube of permeable wall and varying cross-section in presence of slip velocity at the tube wall. In this analysis, we assume that blood is a non- Newtonian biviscous fluid and the blood vessel is a straight, rigid circular tube of varying cross-section. The analytical expressions for the streamlines, wall shear stress and pressure drop are obtained. The influence of slip parameter (due to slip velocity), leakage parameter (due to suction/injection velocity), biviscosity coefficient and Reynolds number (i.e. low Reynolds number only) on the streamlines, wall shear stress and pressure drop are also shown graphically.

2. MATHEMATICAL FORMULATION

The pulsatile motion of an incompressible non-Newtonian biviscous fluid in an axisymmetric rigid circular tube of varying cross-section and permeable wall with slip flow is considered. We consider cylindrical polar coordinate system (r, ɵ, z) such that ɵ = 0 represents the axisymmetry for the tube. Then, the radius of the tube r = R (z) is given by

$$R(z) = R_0\left\{1+\varepsilon S\left(\frac{\varepsilon z}{R_0}\right)\right\} \text{ with } S(0)=1 \qquad (1)$$

where, $\epsilon = R_0/L(<<1)$ is the tube wall slope parameter, L is the characteristic length of the tube and R_0 is the tube radius at $z = 0$. It can be noted that $\epsilon = 0$ gives the case of tube with uniform radius. Again, we assume that the radius of the tube varies slowly along the axial direction so that the velocity depends on r and z only. Now the equations, which govern the pulsatile flow of an incompressible non- Newtonian fluid obeying biviscosity model, in an axisymmetric circular tube can be written as follows:

Continuity equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}=0 \qquad (2)$$

Momentum equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+ \upsilon_B\left(1+b^{-1}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right] \qquad (3)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+ \upsilon_B\left(1+b^{-1}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2}\right], \qquad (4)$$

where (u, o, w) are the velocities components in (r, ɵ, z) directions, t is the time, P is the pressure, υ_B is the kinematic coefficient of viscosity, ρ is the fluid density and b is the upper limit of the apparent viscosity coefficient.

To consider the permeability effect of the tube wall, we prescribe the suction/injection velocity of the fluid at the tube wall to consist of a steady part and an oscillatory part. Thus, the normal component of the fluid velocity at the tube wall is given by:

$$u - \frac{dR}{dz}w = v_s\left(1 + \delta e^{int}\right)\left\{1 + \left(\frac{dR}{dz}\right)^2\right\}^{\frac{1}{2}} \quad \text{at } r = R(z), \tag{5}$$

where, v_s is the steady state suction/injection velocity, δ is the ratio of the amplitudes of the oscillatory and steady parts of the suction/injection velocity and n is the frequency of the oscillation.
The slip equation on the boundary is:

$$w + \frac{dR}{dz}u = u_0 \quad \text{at } r = R(z) \tag{6}$$

i.e., the tangential velocity is non-zero at the wall, where u_0 is the slip parameter.
The axisymmetry of the flow gives:

$$\frac{\partial w}{\partial r} = 0 \text{ and } u = 0 \text{ at } r = 0 \tag{7}$$

Again, the flux at the initial cross-section (i.e., z = 0) is assumed to be in phase with the suction/injection velocity and is taken as

$$Q = Q_s\left(1 + \delta e^{int}\right) \text{ at } z = 0, \tag{8}$$

where, Q_S is the steady state flux at the initial cross-section.
We introduce stream function $\psi(r, z)$ by:

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z} \text{ and } w = \frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{9}$$

Eliminating P (i.e. pressure) from (3) and (4) and using (9), we get:

$$\begin{aligned}&\frac{\partial\Omega}{\partial t} + \left\{\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\left(\frac{\Omega}{r}\right) - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{\Omega}{r}\right)\right\}\\ &= \upsilon_B\left(1 + b^{-1}\right)\left\{\frac{\partial^2\Omega}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\Omega)\right)\right\},\end{aligned} \tag{10}$$

$$\text{where,} \quad \Omega = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right). \tag{11}$$

The boundary Equations (6) and (7) in terms of ψ can be written as follows:

$$\left.\begin{aligned}&\frac{\partial\psi}{\partial r} - \frac{dR}{dz}\frac{\partial\psi}{\partial z} = ru_0 \quad \text{at } r = R(z)\\ &\psi = 0,\ \frac{1}{r}\frac{\partial\psi}{\partial z} = 0,\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = 0 \ \text{ as } r \to 0\end{aligned}\right\} \tag{12}$$

The equation of continuity (2) along with the Equations (5) and (8) for axisymmetric flow gives:

$$\begin{aligned}\psi &= \frac{1}{2\pi}\left(1 + \delta e^{int}\right)\\ &\times\left[Q_s - 2\pi v_s\int_0^z R(\xi)\left\{1 + \left(\frac{dR}{d\xi}\right)^2\right\}^{\frac{1}{2}} d\xi\right]\\ &\text{at } r = R(z).\end{aligned} \tag{13}$$

For convenience, the following dimensionless variables

$$\begin{aligned}&z' = \frac{\varepsilon z}{R_0},\ r' = \frac{r}{R_0},\ t' = nt,\ \psi' = \frac{2\pi\psi}{Q_s},\\ &\omega = 2\pi R_0^3\frac{\Omega}{Q_s},\ p = \frac{2\pi R_0^3 P}{\rho\upsilon_B Q_s}\end{aligned} \tag{14}$$

and the dimensionless parameters is introduced.

$$R_e = \frac{Q_s}{2\pi\upsilon_B R_0},\ \alpha^2 = \frac{nR_0^2}{\upsilon_B},\ v_s' = \frac{2\pi R_0^2}{\varepsilon Q_s}v_s \tag{15}$$

Equations (10), (11), (12) and (13) can be written (after dropping the primes) in non-dimensional form as

$$\begin{aligned}&\alpha^2\frac{\partial\omega}{\partial t} + \varepsilon R_e\left\{\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\omega}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{\omega}{r}\right)\right\}\\ &= \left(1 + b^{-1}\right)\left\{\varepsilon^2\frac{\partial^2\omega}{\partial z^2} + \frac{\partial^2\omega}{\partial r^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} - \frac{\omega}{r^2}\right\},\end{aligned} \tag{16}$$

$$\omega = \frac{1}{r}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \varepsilon^2\frac{\partial^2\psi}{\partial z^2}\right), \tag{17}$$

$$\left.\begin{aligned}&\frac{\partial\psi}{\partial r} - \varepsilon^2\frac{dS}{dz}\frac{\partial\psi}{\partial z} = ru_0,\\ &\psi = \left(1 + \delta e^{int}\right)\left[1 - v_s\int_0^z G(\xi)d\xi\right]\end{aligned}\right\} \text{ at } r = S(z) \tag{18}$$

$$\psi = 0,\ \frac{\partial \psi}{\partial z} = 0,\ \ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = 0 \text{ as } r \to 0, \quad (19)$$

where, Re is the Reynolds number of entrance flow, α is Womersle`s parameter, ν_s is the leakage parameter and

$$G(z) = S(z)\left\{1 + \varepsilon^2\left(\frac{dS}{dz}\right)^2\right\}^{\frac{1}{2}}.$$

In equation (18), it can be noted that $\delta=0$ and $\nu_s=0$ indicates the steady flow and the impermeability of the tube wall respectively.

3. SOLUTIONS

We assume that the pulsatile flow consists of the steady part and the oscillatory part of small amplitude of oscillation δ such that the terms of the order δ^2 can be neglected (i.e. $\delta<<1$)

Therefore, we seek the solutions of (16) to (19) in the following form:

$$\left.\begin{aligned}\omega &= \left(\omega_{00} + \delta e^{it}\omega_{01}\right) + \varepsilon\left(\omega_{10} + \delta e^{it}\omega_{11}\right) + o\left(\varepsilon^2, \delta^2\right),\\ \psi &= \left(\psi_{00} + \delta e^{it}\psi_{01}\right) + \varepsilon\left(\psi_{10} + \delta e^{it}\psi_{11}\right) + o\left(\varepsilon^2, \delta^2\right)\end{aligned}\right\} \quad (20)$$

Here we restrict the analysis for low Reynolds number flows because the exchange of fluid takes place only in the blood capillaries where the Reynolds number of blood is very low (0.02 - 12). Thus, using the perturbation scheme (20) for ω and ψ in equations (16) to (19) and then collecting the coefficients of e^{it} and of equal power of ϵ, we get the following equations and boundary conditions:

(i) Zeroth order steady part:

$$D^2\omega_{00} = 0, \quad (21a)$$

$$\omega_{00} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_{00}}{\partial r}\right), \quad (21b)$$

$$\frac{\partial \psi_{00}}{\partial r} = r u_0,\ \psi_{00} = 1 - \nu_s F(z) \text{ at } r = S(z), \quad (21c)$$

$$\psi_{00} = 0,\ \frac{\partial \psi_{00}}{\partial z} = 0,\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_{00}}{\partial r}\right) = 0 \text{ as } r \to 0, \quad (21d)$$

where,

$$D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} \text{ and } F(z) = \int_0^z S(\xi)\, d\xi.$$

(ii) Zeroth order oscillatory part:

$$D^2\omega_{01} = \alpha_1{}^2\omega_{01}, \quad (22a)$$

$$\omega_{01} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_{01}}{\partial r}\right), \quad (22b)$$

$$\frac{\partial \psi_{01}}{\partial r} = 0,\ \psi_{01} = 1 - \nu_s F(z) \text{ at } r = S(z), \quad (22c)$$

$$\psi_{01} = 0,\ \frac{\partial \psi_{01}}{\partial z} = 0,\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_{01}}{\partial r}\right) = 0 \text{ as } r \to 0, \quad (22d)$$

where, $\alpha_1 = \dfrac{\sqrt{i\alpha^2}}{1 + b^{-1}}$.

(iii) First order steady part:

$$D^2\omega_{10} = R_{e_1}\left\{\frac{1}{r}\frac{\partial \psi_{00}}{\partial r}\frac{\partial \omega_{00}}{\partial z} - \frac{\partial \psi_{00}}{\partial z}\frac{\partial}{\partial r}\left(\frac{\omega_{00}}{r}\right)\right\}, \quad (23a)$$

$$\omega_{10} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_{10}}{\partial r}\right), \quad (23b)$$

$$\frac{\partial \psi_{10}}{\partial r} = 0,\ \psi_{10} = 0 \text{ at } r = S(z), \quad (23c)$$

$$\psi_{10} = 0,\ \frac{\partial \psi_{10}}{\partial z} = 0,\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_{10}}{\partial r}\right) = 0 \text{ as } r \to 0, \quad (23d)$$

where, $R_{e_1} = \dfrac{R_e}{1 + b^{-1}}$.

(iv) First order oscillatory part:

$$D^2\omega_{11} - \alpha_1{}^2\omega_{11} = R_{e_1}\left[\frac{1}{r}\left\{\frac{\partial \psi_{00}}{\partial r}\frac{\partial \omega_{01}}{\partial z} + \frac{\partial \psi_{01}}{\partial r}\frac{\partial \omega_{00}}{\partial z}\right\} - \left\{\frac{\partial \psi_{01}}{\partial z}\frac{\partial}{\partial r}\left(\frac{\omega_{00}}{r}\right) + \frac{\partial \psi_{00}}{\partial z}\frac{\partial}{\partial r}\left(\frac{\omega_{01}}{r}\right)\right\}\right] \quad (24a)$$

$$\omega_{11} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_{11}}{\partial r}\right), \quad (24b)$$

$$\frac{\partial \psi_{11}}{\partial r}=0,\ \psi_{11}=0 \text{ at } r=S(z), \tag{24c}$$

$$\psi_{11}=0,\ \frac{\partial \psi_{11}}{\partial z}=0,\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_{11}}{\partial r}\right)=0 \text{ as } r\to 0. \tag{24d}$$

The Equations (21a, b) and (22a, b) are solved along with the corresponding boundary conditions to give the zeroth order of ω and ψ as:

$$\omega_{00}=-\frac{8}{S^4}\{1-v_sF(z)\}r, \tag{25}$$

$$\psi_{00}=\frac{1}{S^4}\{1-v_sF(z)\}\{2r^2(S^2+g_0u_0) -r^4-2S^2g_0u_0\} \tag{26}$$

$$\omega_{01}=-2\alpha_1\{1-v_sF(z)\}\frac{I_1(\alpha_1 r)}{S^2I_2(\alpha_1 S)}, \tag{27}$$

$$\psi_{01}=\{1-v_sF(z)\}[\alpha_1 rI_0(\alpha_1 S)-2I_1(\alpha_1 r)] \times\frac{r}{\alpha_1 S^2I_2(\alpha_1 S)}, \tag{28}$$

where

$$g_0=\frac{S^4}{4\{1-v_sF(z)\}} \text{ and } I_0(x),\ I_1(x),\ I_2(x)$$

are modified Bessel functions of order 0, 1, 2 respectively.

Using Equations (25) to (28), we solve Equations (23) and (24) for the first order components of ω and ψ. Then the results are obtained as follows:

$$\omega_{10}=-2\frac{R_{e_1}}{S^8}\{1-v_sF(z)\}(g_1+4g_2) \times\left\{\left(S^2+\frac{3}{2}g_0u_0\right)rS^2-2(S^2+g_0u_0)r^3+\frac{2}{3}r^5\right\}, \tag{29}$$

$$\psi_{10}=\frac{1}{36}\frac{R_{e_1}}{S^8}\{1-v_sF(z)\}(g_1+4g_2)\times \left\{(4S^2+9g_0u_0)r^2S^4-9\left(S^2+\frac{3}{2}g_0u_0\right)r^4S^2 +6(S^2+g_0u_0)r^6-r^8-\frac{3}{2}g_0u_0S^6\right\} \tag{30}$$

$$\omega_{11}=\frac{R_{e_1}}{\alpha_1^2S^9I_2(\alpha_1 S)}\{1-v_sF(z)\} \times\{T_1rI_0(\alpha_1 r)-T_2r^2I_1(\alpha_1 r)-T_3r^3I_2(\alpha_1 r) +T_4r^4I_1(\alpha_1 r)-8T_6r+T_7I_1(\alpha_1 r)\}, \tag{31}$$

$$\psi_{11}=\frac{R_{e_1}}{\alpha_1^4S^9I_2(\alpha_1 S)}\{1-v_sF(z)\} \times\{T_1r^2I_2(\alpha_1 r)-T_2r^3I_3(\alpha_1 r)-T_3r^4I_4(\alpha_1 r) +T_4r^5I_3(\alpha_1 r)-\alpha_1^2T_6r^4-T_7rI_1(\alpha_1 r)-T_8r^2\}, \tag{32}$$

where, I_m (m = 3,4) is the modified Bessel functions, g_1, g_2 and T_i (i= 1 to 8) are functions of S(z) and are given by:

$$g_1=v_sS(z),$$

$$g_2=\frac{1}{S}\{1-v_sF(z)\}\frac{dS}{dz},$$

$$T_1=4\alpha_1^2S^5\left[1+g_1+\frac{\alpha_1 SI_1(\alpha_1 S)}{I_2(\alpha_1 S)}g_2\right],$$

$$T_2=\alpha_1 S^3\left[\left(4+\alpha_1^2(S^2+g_0u_0)\right)g_1+2\left(8+(S^2+g_0u_0)\right)g_2\right],$$

$$T_3=\frac{1}{3}\alpha_1^2S^3\left[7g_1+4\left\{3+\frac{\alpha_1 SI_1(\alpha_1 S)}{I_2(\alpha_1 S)}\right\}g_2\right],$$

$$T_4=\frac{1}{4}\alpha_1^3S^3(g_1+4g_2),$$

$$T_5=\frac{2}{3}\alpha_1^2S^3\left[5g_1+2\left\{6+\frac{\alpha_1 SI_1(\alpha_1 S)}{I_2(\alpha_1 S)}\right\}g_2\right],$$

$$T_6=2S^3(g_1+4g_2)I_0(\alpha_1 S),$$

$$T_7=\frac{1}{\alpha_1 I_2(\alpha_1 S)}\left[T_1\{\alpha_1 SI_1(\alpha_1 S)-2I_2(\alpha_1 S)\}-ST_2 \{\alpha_1 SI_2(\alpha_1 S)-2I_3(\alpha_1 S)\}-S^2T_5\{\alpha_1 SI_3(\alpha_1 S)- 2I_4(\alpha_1 S)\}+\alpha_1 S^4T_4I_4(\alpha_1 S)-2\alpha_1^2(S^2+g_0u_0)T_6\right]$$

$$T_8=\frac{1}{I_2(\alpha_1 S)}\Big[I_0(\alpha_1 S)\times\{I_2(\alpha_1 S)T_1-SI_3(\alpha_1 S)T_2+ S^3I_3(\alpha_1 S)T_4-S^2I_4(\alpha_1 S)T_5-\alpha_1^2(S^2+g_0u_0)T_6 -I_1(\alpha_1 S)\Big\{I_1(\alpha_1 S)T_1-SI_2(\alpha_1 S)T_2+\frac{S^2}{\alpha_1}\{\alpha_1 SI_2 (\alpha_1 S)+2I_3(\alpha_1 S)\}T_4-S^2I_3(\alpha_1 S)T_5-4\alpha_1 ST_6\Big\}\ \Big]$$

3.1. Wall Shear Stress:

The shear stress

$$\tau'_w = \frac{\left[\sigma_{zr}\left\{1-\left(\frac{dR}{dz}\right)^2\right\}+\left(\sigma_{rr}-\sigma_{zz}\right)\frac{dR}{dz}\right]}{\left\{1+\left(\frac{dR}{dz}\right)^2\right\}}$$

where,

$$\sigma_{zr} = \mu\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right) \text{ and } \sigma_{rr}-\sigma_{zz} = -2\mu\left(\frac{\partial w}{\partial z}-\frac{\partial u}{\partial r}\right)$$

are calculated at $r = R(z)$.

Then, using the boundary conditions at r = S(z) and Equations (9) and (11), we obtain the dimensionless wall shear stress τ_w in the following form:

$$\begin{aligned}\tau_w &= \frac{2\pi R_0^3}{\mu Q_s}\tau'_w \\ &= \left(\omega_{00}+\delta e^{it}\omega_{01}\right)+\varepsilon\left(\omega_{10}+\delta e^{it}\omega_{11}\right) \\ &+o\left(\varepsilon^2,\delta^2\right) \text{ at } r = S(z). \\ &= -\frac{8\{1-v_s F(z)\}}{S^3}\left[1+\delta e^{it}\left(\frac{\alpha_1 S I_1(\alpha_1 S)}{4I_2(\alpha_1 S)}\right)\right. \quad (33) \\ &-\varepsilon R_{e_1}\left\{\frac{(g_1+4g_2)}{24S^2}\left(2S^2+3g_0u_0\right)-\frac{\delta e^{it}}{8\alpha_1^2 S^6 I_2(\alpha_1 S)}\right. \\ &\left\{T_1 S I_0(\alpha_1 S)-T_2 S^2 I_1(\alpha_1 S)-T_3 S^3 I_2(\alpha_1 S)\right. \\ &\left.\left.\left.+T_4 S^4 I_1(\alpha_1 S)-8T_6 S+T_7 I_1(\alpha_1 S)\right\}\right]\right\}+o\left(\varepsilon^2,\delta^2\right)\end{aligned}$$

3.2. Pressure Drop Using the Equations (3), (4) and the non-dimensionalizing Equations (14), (15) we obtain the non-dimensionalized pressure as:

$$p = \left(p_{00}+\delta e^{it}p_{01}\right)+\varepsilon\left(p_{10}+\delta e^{it}p_{11}\right)+o\left(\varepsilon^2,\delta^2\right).$$

Thus, the equations governing pressure components can be written as:

$$\frac{\partial p_{00}}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\omega_{00}\right), \quad (34)$$

$$\frac{\partial p_{01}}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\omega_{01}\right)-\frac{\alpha_1^2}{r}\frac{\partial\psi_{01}}{\partial r}, \quad (35)$$

$$\begin{aligned}\frac{\partial p_{10}}{\partial z} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\omega_{10}\right) \\ &-\frac{R_{e_1}}{r}\left(\frac{1}{r}\frac{\partial\psi_{00}}{\partial r}\frac{\partial^2\psi_{00}}{\partial r\partial z}-\omega_{00}\frac{\partial\psi_{00}}{\partial z}\right),\end{aligned} \quad (36)$$

$$\begin{aligned}\frac{\partial p_{11}}{\partial z} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\omega_{11}\right)-\frac{\alpha_1^2}{r}\frac{\partial\psi_{11}}{\partial r}-\frac{R_{e_1}}{r} \\ &\left[\left\{\frac{1}{r}\left(\frac{\partial\psi_{00}}{\partial r}\frac{\partial^2\psi_{01}}{\partial r\partial z}-\frac{\partial\psi_{01}}{\partial r}\frac{\partial^2\psi_{00}}{\partial r\partial z}\right)\right\}\right. \\ &\left.-\omega_{00}\frac{\partial\psi_{01}}{\partial z}+\omega_{01}\frac{\partial\psi_{00}}{\partial z}\right]\end{aligned} \quad (37)$$

with

$$\frac{\partial p_{00}}{\partial r} = \frac{\partial p_{01}}{\partial r} = \frac{\partial p_{10}}{\partial r} = \frac{\partial p_{11}}{\partial r} = 0. \quad (38)$$

The Equation (38) indicates that the pressure components are independent of r; hence, Equations (34) to (37) are integrated to give the pressure drop $\Delta p(z) = p(0) - p(z)$ up to the first order as follows:

$$\begin{aligned}\varepsilon\Delta p &= 16\int_0^z\frac{\{1-v_s F(z)\}}{S^4}dz \\ &+2\alpha_1^2\delta e^{it}\int_0^z\frac{\{1-v_s F(z)\}}{S^2 I_2(\alpha_1 S)}I_0(\alpha_1 S)\,dz \\ &-4\varepsilon R_{e_1}\left[\int_0^z\frac{\{1-v_s F(z)\}}{S^4}\times\left\{(3g_1+4g_2)+\right.\right. \\ &\left.\left.\frac{g_0u_0}{2S^4}(g_1+4g_2)\times\left(5S^2+8g_0u_0\right)\right\}dz\right. \\ &\left.+2\delta e^{it}\int_0^z\left[\begin{array}{l}\{1-v_s F(z)\}T_9 \\ -\frac{4v_s}{\alpha_1^2 S}(g_1+4g_2)\end{array}\right]\frac{dz}{S^4 I_2(\alpha_1 S)}\right]+o\left(\varepsilon^2,\delta^2\right)\end{aligned}$$

where,

$$\begin{aligned}T_9 &= 2I_0(\alpha_1 S)g_1+\left\{2I_0(\alpha_1 S)-\right. \\ &\left.\alpha_1 S I_1(\alpha_1 S)+\frac{\alpha_1 S I_0(\alpha_1 S)}{I_2(\alpha_1 S)}\right\}g_2+\frac{T_8}{4\alpha_1^2 S^5}.\end{aligned}$$

4. NUMERICAL RESULTS AND DISCUSSION

In the above mathematical analysis, the expressions of the flow variables ψ, τ_w, and Δp depend upon the following non-dimensional parameters:

slip parameter u_0,leakage parameter υ_s, upper limit of the apparent viscosity coefficient b, Reynolds number of entrance flow R_e.

Due to the presence of complex parameter α_1, the results obtained for ψ, τ_w, and Δp appear in the complex form. From the physical point of view, we consider only the real part of the expressions and plot the values against z. The results are obtained by taking:

$$\varepsilon = 0.1,\ \delta = 0.1,\ \alpha = 1,\ t = \frac{\pi}{4} \text{ and}$$

$$v_s = 0.2, -0.2 \ \left(\text{for suction/injection} \right)$$

for the following tube geometries:

(i) Sinusoidal tube
i.e., $S(z) = 1 + 0.2\sin(2\pi z)$,

(ii) Locally constricted tube
i.e., $S(z) = \dfrac{2 - \exp\left\{-(z - 0.5)^2\right\}}{2 - \exp(-0.25)}$.

The analytical results obtained in this work are more generalized form of Prasad et al. [9] and can be taken as a limiting case by taking $b \to \infty$ and $u_0 \to 0$.

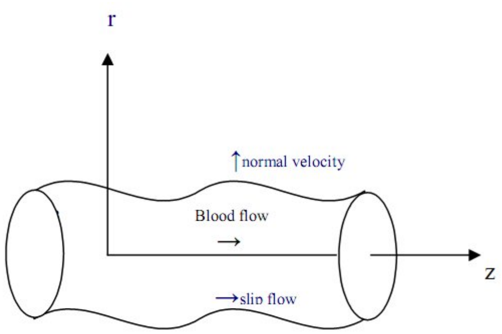

Fig. 1. Blood flow through a circular tube of varying cross-section

4.1. Streamlines The real part of dimensionless streamlines ψ is plotted for different values of slip parameter u_0, Reynolds number R_e and upper limit of apparent viscosity b in Figures 2 and 3 (for sinusoidal tube) and Figures 4 and 5 (for locally constricted tube). It is observed that the deviation of flow increases with an increase in u_0, R_e and b for suction and injection velocity. It means that either for suction or injection velocity, small values of u_0, R_e or b give gentle flow whereas higher values of u_0, R_e and b give reckless flow. But due to the presence of suction velocity, the deviation of flow is less than that for the injection velocity.

4.2. Wall Shear Stress The characteristic of the real part of non-dimensional wall shear stress τ_w is displayed through Figures 6, 7 (for sinusoidal tube), 8 and 9 (for locally constricted tube). Figures 6 and 7 show that with an increase in u_0, R_e and b, the value of τ_w decreases in the converging region and increases in the diverging region of the tube. From Figures 8 and 9, it is seen that the similar results occur for a locally constricted tube.

4.3. Pressure Drop The effects of different parameters on the real part of dimensionless pressure drop Δp are indicated graphically through Figures 10 and 11 (for sinusoidal tube) and 12, 13 (for locally constricted tube). Figures 10 and 11 show that at any cross-section of the sinusoidal tube, Δp decreases with an increase in u_0 and increases with an increase in b for both suction and injection velocities. But increase in R_e increases Δp for the cross-section $0 \le z < 0.7$ and decreases Δp for $z > 0.7$. It is shown through Figures 12 and 13 depict that at any cross-section of the locally constricted tube, the pressure drop Δp decreases with an increase in u_0, R_e and b for both suction and injection velocities.

5. CONCLUSIONS

In the present study, we considered the effect of slip parameter, leakage parameter (due to suction/injection velocity at the tube wall), Reynolds number and upper limit of apparent viscosity on pulsatile flow of a non-Newtonian incompressible biviscous fluid in a circular tube with varying cross-section (e.g. some organs in

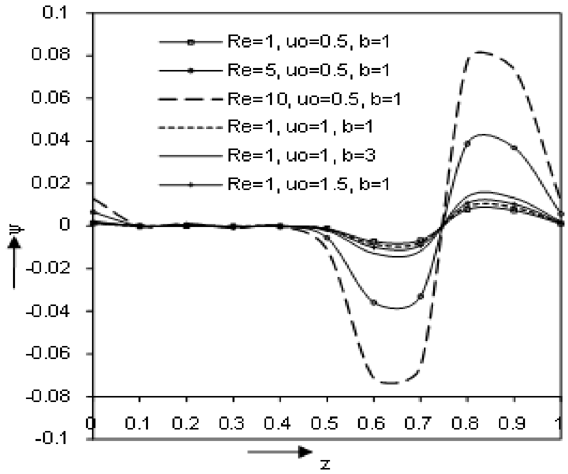

Fig. 2. ψ vs z for sinusoidal tube with suction at the wall

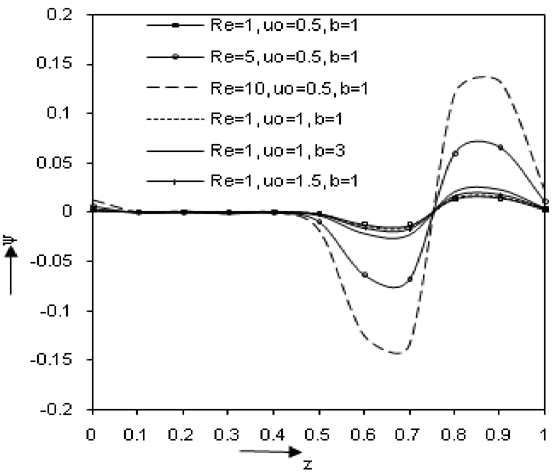

Fig. 3. ψ vs z for sinusiodal tube with injection at the wall

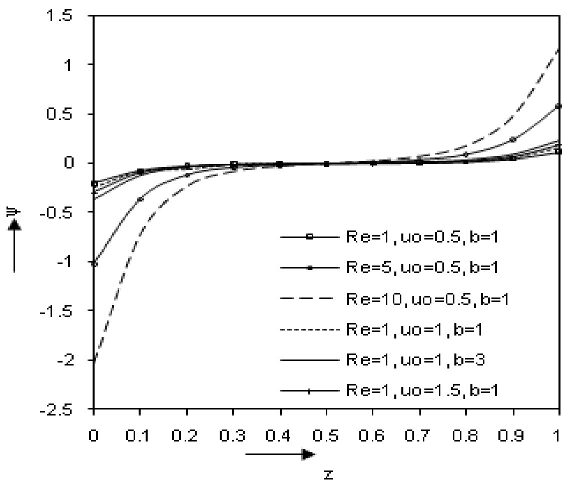

Fig. 4. ψ vs z for locally constricted tube with suction at the wall

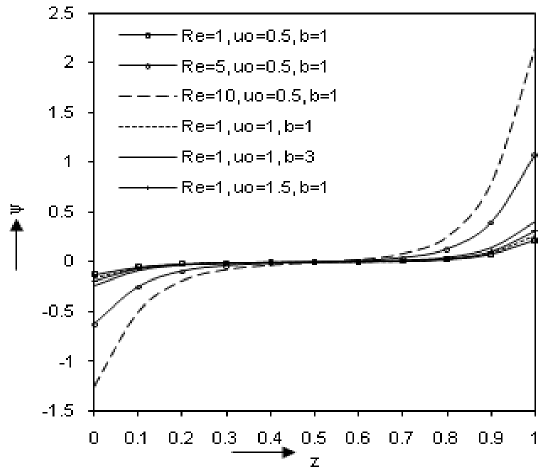

Fig. 5. ψ vs z for locally constricted tube with injection at the wall

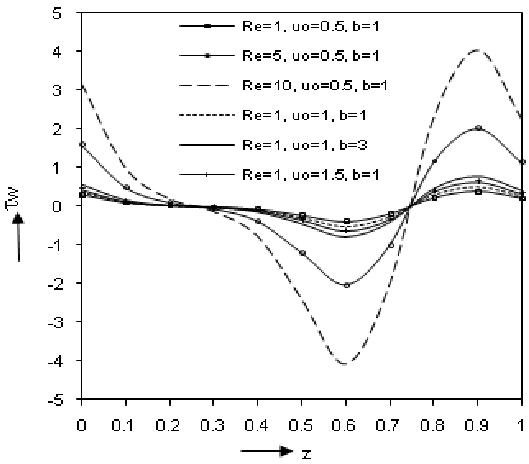

Fig. 6. τw vs z for sinusoidal tube with suction at the wall

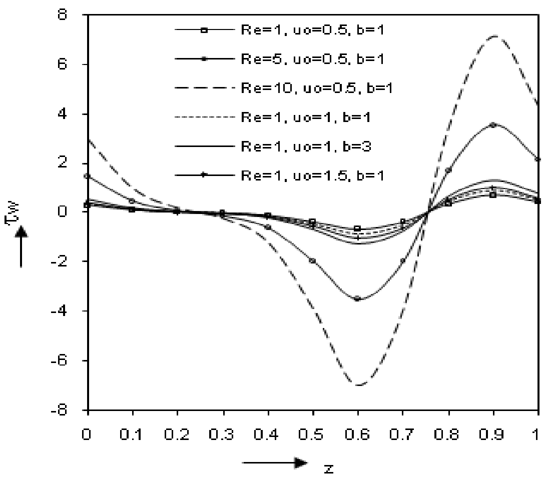

Fig. 7. τw vs z for sinusoidal tube with injection at the wall

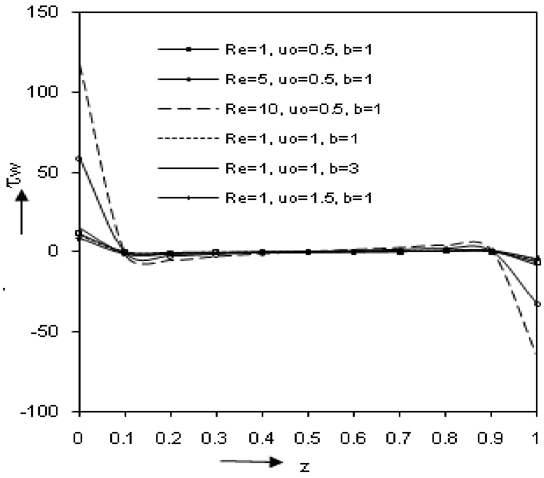

Fig. 8. τw vs z for locally constrited tube with suction at the wall

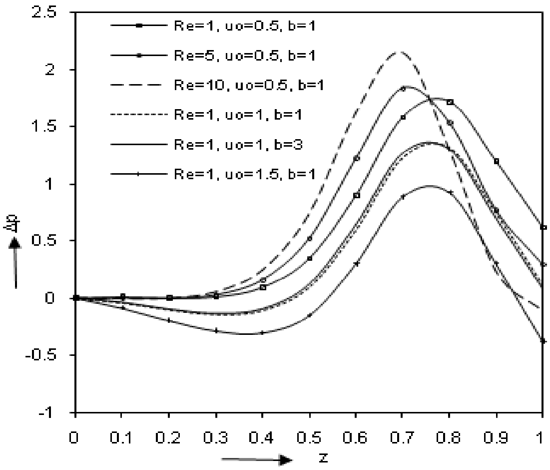

Fig. 11. Δp vs z for sinusoidal tube with injection at the wall

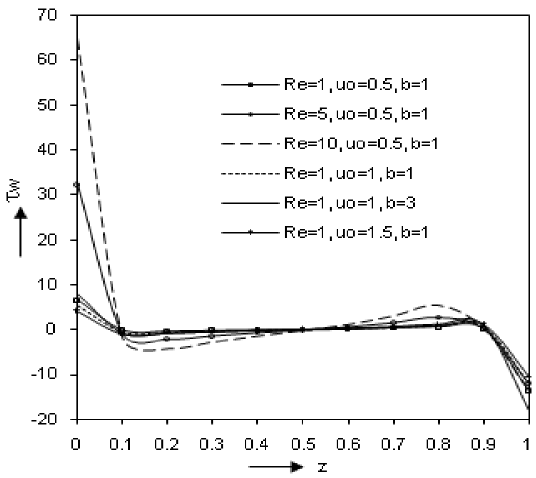

Fig. 9. τw vs z for locally constricted tube with injection at the wall

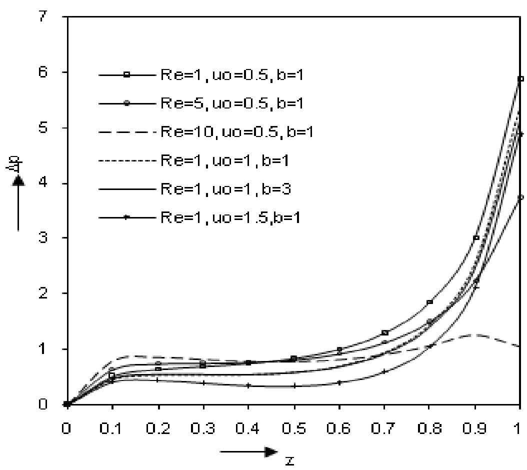

Fig. 12. Δp vs z for locally constricted tube with suction at the wall

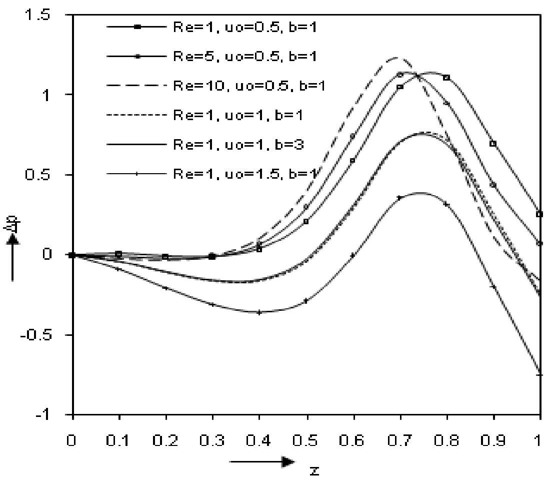

Fig. 10. Δp vs z for sinusoidal tube with suction at the wall

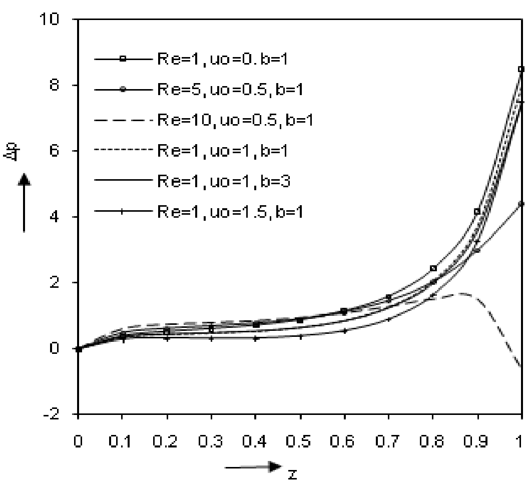

Fig. 13. Δp vs z for locally constricted tube with injection at the wall

human body). This investigation helps us to note that the influence of slip parameter in the pressure drop is much significant and decreases rapidly with increases in slip parameter. The increase in pressure drop indicates the rise in systolic pressure and fall in diastolic pressure, which are very dangerous for heart. It is also to be noted that this presentation help us to draw the flow characteristic of blood and the wall shear stress on the inner wall of capillaries. It illustrates the small blood vessels where suction, injection and slip velocities arise and Reynolds number is very low. So, this investigation may be helpful in various fields of medical science.

6. ACKNOWLEDGEMENT

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