# ANALYTICAL SOLUTION OF THE LAMINAR BOUNDARY LAYER FLOW OVER SEMI-INFINITE FLAT PLATE: VARIABLE SURFACE TEMPERATURE

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**Abstract** In this paper, the problem of forced convection over a horizontal flat plate under condition of variable plate temperature is presented and the homotopy perturbation method (HPM) is employed to compute an approximation to the solution of the system of nonlinear differential equations governing the problem. This paper provides good supplements to the existing literature and generalizes the previous investigation of the problem under condition of fixed plate surface temperature. It is shown that HPM is accurate and reliable with high acceleration in converging.

**Keywords** Homotopy perturbation method (HPM); Nonlinear differential equations; Laminar thermal boundary layer; Forced convection ; Horizontal flat plate; Variable surface temperature

**چکیده** در این مقاله مساله جابجایی اجباری بر روی یک صفحه مسطح افقی در دماهای مختلف صفحه مطرح شده و روش اختلال هموتوپی برای محاسبه جواب تقریبی سیستم معادلات دیفرانسیل غیر خطی به کار گرفته می شود. نتایج به دست آمده از این تحقیق تکمیل کننده متون موجود در این زمینه بوده و بررسی های قبلی در مورد این مساله در شرایط دمای ثابت سطح را عمومیت می بخشد. نتایج نشان می دهد روش اختلال هموتوپی روشی دقیق و قابل اطمینان با دقت بالا در مساله جابجایی می باشد.

## 1. INTRODUCTION

Most of scientific problems are modeled using differential equations such as ordinary differential equations, partial differential equations, fractional differential equations, algebraic differential equations and integro-differential equations. In general, despite a limited number of cases, most of them do not have analytical solution. The equations which govern the boundary layer problems don't have exact solutions and require to be solved by semi-exact analytical or numerical methods. Numerical techniques have their merits and demerits in special cases. In the numerical methods, stability and convergence should be considered so as to avoid divergence or inappropriate results. One of these semi-analytical methods is perturbation method. In this method, a small parameter should be exerted in the equation. Therefore, finding the small parameter and exerting it into the equation are difficulties of this method. Since there are some limitations with the common perturbation method, and also because the basis of the common perturbation method is upon the existence of a small parameter, developing the method for different applications is very difficult[1].

Therefore, many different methods have recently been introduced to eliminate the small parameter, such as artificial parameter method [2,3], homotopy perturbation method [4,5], and variational iteration method [6–8] introduced by He. Homotopy perturbation method is one of these semi-exact analytical methods [9–21]. The applications of this method in different fields of nonlinear equations, integro-differential equations, Laplace transform, fluid mechanics and heat transfer have been studied by Cai [22], Cveticanin [23], El-Shahed [24], Abbasbandy [25], Siddiqui [20,21] and Ganji [26–28]. In this paper, He's homotopy perturbation method is applied to the problem of forced convection over a horizontal flat plate under variable surface temperature condition, to obtain an approximate Solution.

In the flow with inertia effects the equations of motion is nonlinear. Fluid flow around a body can be divided into two parts: a thin layer close to the body which the friction forces play an important role in it and the other is the outer layer which is not considered as friction force. It is supposed that friction turns our problem into a boundary layer problem. Viscosity is one of the main sources of friction forces in a flow. The Reynold's number determines whether a flow is laminar or not. In low *Re* the flow is laminar but in high Re it is not. So the results of flow based on *Re* number will be completely different (Figure 2). Applications of boundary layers are in pipe lines, aerodynamics (designing war planes, spacecrafts, etc) and so on.

**Nomenclature**

| | |
|---|---|
| HPM | homotopy perturbation method |
| NM | numerical method |
| p | parameter of homotopy |
| Pr | Prandtl number |
| Re | Reynolds number |
| T | temperature |
| $T_s$ | temperature imposed on the plate |
| $T_\infty$ | local ambient temperature |
| C | constant |
| *f* | dimensionless function of velocity |
| *u* | velocity component in the x direction |
| *v* | velocity component in the y direction |
| *y* | dimensional vertical coordinate |
| *x* | dimensional horizontal coordinate |

**Greek symbols**

| | |
|---|---|
| ν | kinematic viscosity |
| α | thermal diffusivity |
| θ | dimensionless temperature |
| α | constant |
| η | dimensionless variable similarity solution |

## 2. BASIC IDEAS OF HOMOTOPY PERTURBATION METHOD

The homotopy perturbation method (HPM) is a combination of the classical perturbation technique and homotopy technique. To explain the basic ideas of the HPM for solving nonlinear differential equations, the following nonlinear differential equation is considered:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{1}$$

subject to the boundary condition of:

$$B(u, \partial u / \partial n) = 0, \quad r \in \Gamma \tag{2}$$

where $A$ is a general differential operator, $B$ a boundary operator, $f(r)$ is a known analytical function, $\Gamma$ is the boundary of domain $\Omega$ and $\frac{\partial u}{\partial n}$ **Nomenclature** denotes differentiation along the normal drawn outwards from $\Omega$. The operator $A$ can generally be divided into two parts: a linear part, $L$, and a nonlinear part , $N$ . Eq. (1) therefore can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \tag{3}$$

In this case the nonlinear Eq. (1) has no "small parameter", so the following homotopy can be constructed:

$$H(u, p) = (1-p)[L(u) - L(u_0)] + p[L(u) + N(u) - f(r)] = 0 \tag{4}$$

where $p$ is called homotopy parameter. According to the HPM, the approximate solution of Eq. (4) can be expressed as series of power of $p$ :

$$u(p) = u_0 + pu_1 + p^2u_2 + \cdots \qquad (5\text{-a})$$

$$u = \lim_{p\to 1} u(p) = u_0 + u_1 + u_2 + \cdots \qquad (5\text{-b})$$

When $p \to \infty$, Eq. (4) corresponds to Eq. (1), and Eq. (5b) becomes the approximate solution of Eq. (1).

## 3. GOVERNING EQUATIONS

A uniform flow over a semi-infinite flat plate is shown in Figure1. Surface temperature varies with axial distance x according to the following equation:

$$T_s(x) = T_\infty + C\,x^\alpha\,, \qquad (6)$$

where $C$ and $\alpha$ are constants and $T_\infty$ is free stream temperature [29-35].

Laminar flow starts at the leading edge of a flat plate and continues until a Reynolds number of about 350,000, depending upon the surface roughness and the degree of turbulence (see App. Figure 2).

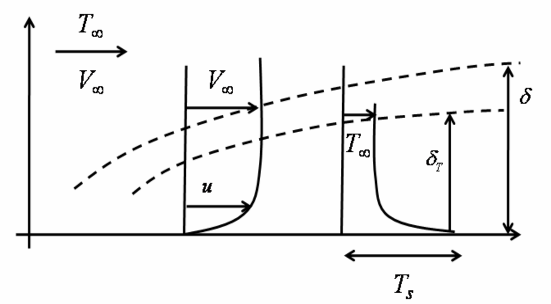


**Figure 1:** Velocity and thermal boundary layers

### 3.1. Summary of boundary layer equations for steady laminar flow

The more general equations for any 2-dimentional flow are given by:

$$\begin{cases} \rho\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) = \nu\dfrac{\partial^2 u}{\partial y^2} - \dfrac{dp_0}{dx} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases} \qquad (7)$$

The energy equation for an incompressible flow field is given by:

$$\rho c_p\left(\frac{\partial T}{\partial t} + \vec{u}\cdot\nabla T\right) = k\nabla^2 T + \dot{q} \qquad (8)$$

where $\frac{\partial T}{\partial t}$ is energy storage term, $\vec{u}\cdot\nabla T$ shows enthalpy convection, $k\nabla^2 T$ presents heat conduction, and $\dot{q}$ is heat generation. The term $\frac{\partial T}{\partial t} + \vec{u}\cdot\nabla T$ is the rate of change of temperature of a fluid particle as it moves in a flow field. In a steady two-dimensional flow field without heat sources, Eq. (8) takes the form:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad (9)$$

Furthermore, in a boundary layer, $\frac{\partial^2 T}{\partial x^2} \quad \frac{\partial^2 T}{\partial y^2}$, so the energy equation is

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (10)$$

In formulating the governing equations for convective heat transfer several simplifying assumptions were made to limit the mathematical complexity. These assumptions are as follow:

(1) Flow is continuous,
(2) Flow is a Newtonian fluid,
(3) Flow is considered in two-dimension,
(4) Negligible changes in kinetic and potential energy and constant properties are considered.

The additional assumptions leading to boundary layer simplifications are:
(5) Slender surface,
(6) High Reynolds number ($Re$ > 100), and high Peclet number ($Pe$ > 100).

Finally, the following additional simplifications are introduced:
(7) Steady state flow,
(8) Laminar flow,
(9) No energy generation $(\dot{q} = 0)$.

The governing boundary layer equations by assuming these conditions and considering constant pressure $(\frac{dp}{dx}=0)$ are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \qquad (11)$$
$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2},$$

the boundary conditions are as follow:

$$u(x,0)=0,\ v(x,0)=0,\ u(x,\infty)=V_\infty,\ u(0,y)=V_\infty \qquad (11\text{-a})$$

$$T(x,0)=T_s=T_\infty+C\,x^{\alpha},\ T(x,\infty)=T_\infty,\ T(0,y)=T_\infty \qquad (11\text{-b})$$

**3.2. Velocity distribution (Blasius similarity solution method)** For this problem, transformation variable η is:

$$\eta(x,y)=y\sqrt{\frac{V_\infty}{\nu x}}=\frac{y}{x}\sqrt{\mathrm{Re}_x} \qquad (12)$$

and the velocity $u(x, y)$ is assumed to depend on η according to the following equation:

$$\frac{u}{V_\infty}=\frac{\partial f}{\partial \eta}, \qquad (13)$$

From continuity Eq. (11), the velocity is obtained:

$$v=-\int\frac{\partial u}{\partial x}dy, \qquad (14)$$

from Eq.(12), dy is:

$$dy=\sqrt{\frac{\nu x}{V_\infty}}\,d\eta \qquad (15)$$

using the chain rule, the derivative $\frac{\partial u}{\partial x}$ is expressed in terms of η, $\frac{\partial u}{\partial x}=\frac{du}{d\eta}\frac{d\eta}{dx}$, and using Eq. (12) and (13) the following equation is obtained:

$$\frac{\partial u}{\partial x}=-\frac{V_\infty}{2x}\eta\frac{d^2 f}{d\eta^2} \qquad (16)$$

Substituting Eq. (15) and (16) into Eq. (14) and rearranging expression gives:

$$\frac{v}{V_\infty}=\frac{1}{2}\sqrt{\frac{\nu}{V_\infty x}}\int\eta\frac{d^2 f}{d\eta^2}d\eta \qquad (17)$$

Integration by parts gives:

$$\frac{v}{V_\infty}=\frac{1}{2}\sqrt{\frac{\nu}{V_\infty x}}\left(\eta\frac{df}{d\eta}-f\right) \qquad (18)$$

In addition to $u, v,$ and $\frac{du}{dx}$, the derivatives $\frac{du}{dy}$ and $\frac{d^2u}{dy^2}$ must be expressed in terms of $\eta$. Using the chain rule and Eq. (12) and (13), we obtain:

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial y}\frac{\partial \eta}{\partial y}=V_\infty\sqrt{\frac{V_\infty}{\nu x}}\frac{d^2 f}{d\eta^2} \qquad (19)$$

$$\frac{\partial^2 u}{\partial y^2}=V_\infty\sqrt{\frac{V_\infty}{\nu x}}\frac{d^3 f}{d\eta^3}$$
(20)

Substituting Eq. (12), (13), and (16)-(20) into momentum Eq. (11) gives:

$$2f'''+f\,f''=0 \qquad (21)$$

Thus, the governing partial differential equations are successfully transformed into an ordinary differential equation. Using Eq. (12)-(18), boundary conditions (11-a) transform into:

$$f'(0)=0,\ f(0)=0,\ f'(\infty)=1 \qquad (22)$$

$$f_0(0)=0,\ f_0'(0)=0,\ f_0'(\infty)=1,\ \theta_0(0)=1,\ \theta_0(\infty)=0 \qquad (33\text{-a})$$

$$p^1:\ \begin{cases}\frac{1}{2}f_0f_0''+f_1'''=0\\ \theta_1''+\frac{1}{2}\Pr f_0\theta_0'-\alpha\Pr f_0'\theta_0=0\end{cases} \qquad (34)$$

**3.3. Temperature distribution (Pohlhausen similarity solution method)** The solution to energy equation is obtained by the method of similarity transformation. A dimensionless temperature $\theta$ is defined as:

$$\theta(x,y)=\frac{T(x,y)-T_\infty}{T_s-T_\infty} \tag{23}$$

To solve energy Eq. (11) using the similarity method, the two independent variables $x$ and $y$ are combined into a single variable, $\eta(x,y)$. For this problem the correct form of the transformation variable $\eta$ is the same as that used in Blasius solution:

$$\eta(x,y)=y\sqrt{\frac{V_\infty}{\nu x}} \tag{24}$$

The solution $\theta(x,y)$ is assumed to depend on $\eta$ so that Eqs. (12)-(18) and (24) gives:

$$\begin{cases} T=T_\infty+(T_s-T_\infty)\theta=T_\infty+Cx^\alpha\theta & (25)\\ \frac{\partial T}{\partial x}=C\alpha x^{\alpha-1}\theta+Cx^\alpha\frac{d\theta}{dx}=C\alpha x^{\alpha-1}+Cx^\alpha\left(-\frac{\eta}{2x}\right)\frac{d\theta}{d\eta} & (26)\\ \frac{\partial T}{\partial y}=Cx^\alpha\frac{d\theta}{dy}=Cx^\alpha\sqrt{\frac{V_\infty}{\nu x}}\frac{d\theta}{d\eta} & (27)\\ \frac{\partial^2 T}{\partial y^2}=Cx^\alpha\frac{V_\infty}{\nu x}\frac{d^2\theta}{d\eta^2} & (28)\end{cases}$$

substituting Eqs.(26)-(28) into energy Eq. (11) simplifies it to:

$$\theta''-\alpha\Pr f'\theta+\frac{1}{2}\Pr f\,\theta'=0 \tag{29}$$

where $\Pr=\frac{\nu}{\alpha}$. Boundary conditions (11-b) become:

$$\theta(0)=1,\ \theta(\infty)=0 \tag{30-a}$$

for $\alpha=0$, the Eq. (29) simplifies to:

$$\theta''+\frac{1}{2}\Pr f\,\theta'=0 \tag{30-b}$$

## 4. APPLYING HPM TO THE PROBLEM

In this section, the HPM was applied to the nonlinear ordinary differential system (21) and (29). According to the HPM, the homotopy of system (21) and (29) were constructed as follows:

$$\begin{cases}(1-p)[f'''-f_0''']+p[2f'''+ff'']=0,\\ (1-p)[\theta''-\theta_0'']+p[\theta''-\alpha\Pr f'\theta+\frac{1}{2}\Pr f\,\theta']=0.\end{cases} \tag{31}$$

$f$ and $\theta$ were considered as:

$$\begin{cases} f=f_0+pf_1+p^2f_2+p^3f_3+\cdots\\ \theta=\theta_0+p\theta_1+p^2\theta_2+p^3\theta_3+\cdots\end{cases} \tag{32}$$

by substituting $f$ and $\theta$ from Eq. (32) into Eq. (31), doing some simplification, and rearranging based on powers of $p$-terms, we have:

$$p^0:\begin{cases} f_0'''=0\\ \theta_0''=0\end{cases} \tag{33}$$

$$f_1(0)=0,\ f_1'(0)=0,\ f_1'(\infty)=0,\ \theta_1(0)=0,\ \theta_1(\infty)=0 \tag{34-a}$$

$$p^2: \begin{cases} f_2'''+\frac{1}{2}f_1f_0''+\frac{1}{2}f_0f_1''=0 \\ \theta_2''+\frac{1}{2}\Pr f_0\theta_1'+\frac{1}{2}\Pr f_1\theta_0'-\alpha \Pr f_1'\theta_0-\alpha \Pr f_0'\theta_1=0 \end{cases} \tag{35}$$

$$f_2(0)=0,\ f_2'(0)=0,\ f_2'(\infty)=0,\ \theta_2(0)=0,\ \theta_2(\infty)=0 \tag{35-a}$$

$$p^3: \begin{cases} f_3'''+\frac{1}{2}f_2f_0''+\frac{1}{2}f_0f_2''+\frac{1}{2}f_1f_1''=0 \\ \theta_3''+\frac{1}{2}\Pr f_0\theta_1'+\frac{1}{2}\Pr f_0\theta_2'+\frac{1}{2}\Pr f_2\theta_0'-\alpha \Pr f_1'\theta_1-\alpha \Pr f_0'\theta_2-\alpha \Pr f_2'\theta_0=0 \end{cases} \tag{36}$$

$$f_3(0)=0,\ f_3'(0)=0,\ f_3'(\infty)=0,\ \theta_3(0)=0,\ \theta_3(\infty)=0 \tag{36-a}$$

*by solving Eqs. (33)–(36) with boundary conditions (29-a)–(32-a), we obtain:*

$$f_0=0.1\eta^2 \tag{37}$$

$$f_1=0.052\eta^2-1.6\times10^{-4}\eta^5 \tag{38}$$

$$f_2=0.020\eta^2-1.7\times10^{-4}\eta^5+5.5\times10^{-7}\eta^8 \tag{39}$$

$$f_3=0.020\eta^2-1.1\times10^{-4}\eta^5+8.510\times10^{-7}\eta^8-1.9\times10^{-9}\eta^{11} \tag{40}$$

$$\theta_0=1-0.2\eta \tag{41}$$

$$\theta_1=(-0.104\Pr-0.417\alpha\Pr)\eta+0.033\alpha\Pr\eta^3+(8.3\times10^{-4}\Pr-0.003\alpha\Pr)\eta^4 \tag{42}$$

$\theta_2$ *and* $\theta_3$ *are calculated similarly*
*Then,*

$$f(\eta)=\sum_{i=0}^{3} f_i = 0.1743\eta^2 - 4.5\times10^{-4}\eta^5 + 1.4\times10^{-6}\eta^8 - 1.9\times10^{-9}\eta^{11} \tag{43}$$

$$\begin{aligned}\theta(\mathrm{Pr},\alpha,\eta)=\sum_{i=0}^{3}\theta_i &= (-0.53\alpha^3\,\mathrm{Pr}^3 + 0.36\alpha^2\,\mathrm{Pr}^2 - 0.40\alpha^2\,\mathrm{Pr}^3 - 0.042\alpha\,\mathrm{Pr}^3 + 0.18\alpha\,\mathrm{Pr}^2 - 0.063\alpha\,\mathrm{Pr}\\ &-0.016\,\mathrm{Pr} + 0.006\,\mathrm{Pr}^2)\eta + 0.007\alpha\,\mathrm{Pr}\,\eta^3 + (2\times10^{-4}\,\mathrm{Pr} + 0.007\alpha^3\,\mathrm{Pr}^3 - 0.007\alpha^2\,\mathrm{Pr}^2 + 4\times1\\ &-3\times10^{-5}\,\mathrm{Pr}^3 + 1.9\times10^{-14}\alpha\,\mathrm{Pr}^2 - 0.001\alpha\,\mathrm{Pr} + 0.002\alpha^2\,\mathrm{Pr}^3 - 0.001\alpha\,\mathrm{Pr}^3)\eta^4 + (2\times10^{-4}\alpha^2\\ &-2\times10^{-4}\alpha\,\mathrm{Pr}^2 - 3\times10^{-5}\alpha\,\mathrm{Pr})\eta^6 + \cdots\end{aligned} \tag{44}$$

For $\mathrm{Pr}=1,\ \alpha=0,\ and\ \alpha=0.4$ Eq. (40) gives:

$$\theta(1,0,\eta)=1-0.349\eta+0.002\eta^4-1.1\times10^{-5}\eta^7+2.1\times10^{-8}\eta^{10} \tag{45}$$

$$\theta(1,0.4,\eta)=1-0.45\eta+0.023\eta^3-0.002\times10^{-3}\eta^4-8.6\times10^{-5}\eta^6+8.8\times10^{6}\eta^7+1.4\times10^{7}\eta^9-1.1\times10^{8}\eta^{10} \tag{46}$$

## 5. CONCLUSION

Numerical results are shown in Tables 1 and 2. Also the obtained results of the Homotopy perturbation and numerical methods have been compared in Figures 3 and 4. Clearly, there is a good agreement between the results obtained by using both methods. The results show that the HPM employed to solve the boundary layer equations of the laminar flow over flat plate at variable temperatures provides excellent approximation to the solution of this nonlinear system with high accuracy and acceleration in converging. Contours of dimensionless function of temperature, $\theta(x,y)$, are illustrated in Figures 5-9 for different values of parameters $\mathrm{Pr}$ and $\alpha$. The parameters $\mathrm{Pr}$ and $\alpha$ in Figures 7 and 8 have been set for pure water at initial temperature of $33^\circ$ C.Numerical results are shown in Tables 1 and 2. Also the obtained results of the Homotopy perturbation and numerical methods have been compared in Figures 3 and 4. Clearly, there is a good agreement between the results obtained by using both methods. The results show that the HPM employed to solve the boundary layer equations of the laminar flow over flat plate at variable temperatures provides excellent approximation to the solution of this nonlinear system with high accuracy and acceleration in converging. Contours of dimensionless function of temperature, $\theta(x,y)$, are illustrated in Figures 5-9 for different values of parameters $\mathrm{Pr}$ and $\alpha$. The parameters $\mathrm{Pr}$ and $\alpha$ in Figures 7 and 8 have been set for pure water at initial temperature of $33^\circ$ C.

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