

A COMPUTATIONAL STUDY OF OXYGEN TRANSPORT IN THE BODY OF LIVING ORGANISM

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Abstract Oxygen is an essential part of the living organism. It is transported from blood to the body tissue by the systematic circulation and large part of it is stored in the blood flowing in capillaries. In this work we discuss a mathematical model for oxygen transport in tissues. The governing equations are established assuming that the blood is flowing along a co-axial cylindrical capillary inside the tissue and has a constant partial pressure of oxygen. We solve the governing partial differential equations using finite element techniques. The main object of the present work is to investigate the effects of various assumptions such as neglecting axial diffusion, neglecting the effect of facilitated myoglobin diffusion etc.

Keywords: Axial diffusion, Hemoglobin, Myoglobin, Co-axial cylindrical capillary

چکیده بخش عمده ای از بافت زنده را اکسیژن تشکیل می دهد. اکسیژن توسط یک چرخش نظام یافته، از خون به بافت های بدن منتقل و بخش عمده ای از آن در مویرگ ها ذخیره می شود. در کار حاضر یک مدل ریاضی برای جا به جایی اکسیژن در بافت ارائه شده است. معادلات حاکم بر این مدل، با فرض حرکت خون در راستای لوله های موئین استوانه ای شکل هم محور و ثابت بودن فشار جزئی اکسیژن نوشته شده است. معادلات دیفرانسیل جزئی به روش المان محدود حل شده است. مهم ترین هدف کار حاضر بررسی تأثیر فرضیات مختلف نظیر حذف نفوذ محوری، حذف اثر نفوذ تسهیل شده میوگلوبین و غیره می باشد.

1. INTRODUCTION

Blood consists of many components such as (i) the red blood cells, which contain the protein molecule that acts as the carrier of oxygen and carbon dioxide in the human system, (ii) the white blood cells which ingest and destroy harmful bacteria's

and dead cells and behave as the defense forces of the body against infection and injury, (iii) platelets that help in the clotting of blood and (iv) plasma which is the liquid part of the blood and contains several salts, glucose, amino acids, proteins, hormones etc. Here we are considering our study only on transport of oxygen in the blood. Oxygen

is transported to the tissues by the systemic circulation. During its transport in capillary region a small fraction of oxygen is dissolved in plasma. While in tissue region it is driven by pressure gradient. Oxygen stored in tissue may be dissolved or chemically bound to myoglobin. A large amount of oxygen stored in the blood flowing into the capillaries is chemically bound to hemoglobin. As the oxygen is transported from blood to tissue, therefore it involves convection, diffusion and reaction processes.

Schubert and Zhang [7] reported about the importance of axial diffusion in the experimental data through a model consisting of cylindrical capillary with blood flowing through it, and surrounded by a co-axial cylindrical tissue compartment, despite the common assumption of neglecting diffusion in the direction of blood flow, Sharan et al [4] considered a single capillary surrounded by a co-axial cylindrical tissue compartment. In this capillary, diffusion is in both axial and radial directions. Many researchers have investigated the solution of the governing equations using various simplifications. Murray [2] considered the one-dimensional transport of oxygen through a solution containing either hemoglobin or myoglobin. Reaction terms, between oxygen and myoglobin in the tissue and between oxygen and hemoglobin in the blood, are taken of non-linear nature. This model considers diffusion in one dimension and reaction between oxygen and either hemoglobin or myoglobin molecule.

2. OUR MODEL

The geometry of the present mathematical model is explained by figure 1. Here blood flowing into the capillary from the artery is named as arterial blood. The equations governing oxygen transport in the capillary are:

$$a_p D_p \nabla^2 p = v a_p \frac{\partial p}{\partial z} - A_c \quad (1)$$

$$C_h D_h \nabla^2 H = v C_h \frac{\partial H}{\partial z} + A_c \quad (2)$$

Here A_c is the reaction term of hemoglobin with oxygen. As in Kapur [3]

$$A_c = \frac{f(p)}{1 + f(p)} \quad (3)$$

Using the dissociation curve given by Kelman [1],

$$f(p) = \begin{cases} \frac{a_1 p + a_2 p^2 + a_3 p^3 + p^4}{a_4 + a_5 p + a_6 p^2 + a_7 p^3 + p^4} & p \leq 12 \\ .003683p + .000584p^2 & p > 12 \end{cases} \quad (4)$$

The constants used in equation (4) are given by

$$a_1 = 8.5322289 \times 10^3$$

$$a_2 = 2.1214010 \times 10^3$$

$$a_3 = -6.7073989 \times 10^2$$

$$a_4 = 9.3596087 \times 10^5$$

$$a_5 = -3.1346258 \times 10^4$$

$$a_6 = 2.3961674 \times 10^3$$

$$a_7 = -6.7104406 \times 10^3$$

When oxygen pressure and hemoglobin saturation are in equilibrium then reaction term (3) in the governing equations will vanishes.

Governing equations in tissue region are:

$$a_t D_t \nabla^2 p = -A_t + Q \quad (5)$$

$$C_m D_m \nabla^2 M = -A_t \quad (6)$$

Here ' A_t ' is reaction term between oxygen and myoglobin and is given by

$$A_t = u_m C_m M - a_t u_m^1 C_m (1 - M) p \quad (7)$$

and in equilibrium, the dissociation curve for myoglobin saturation is

$$g(p) = \frac{u'_m a_t p}{u'_m a_t p + u_m} \quad (8)$$

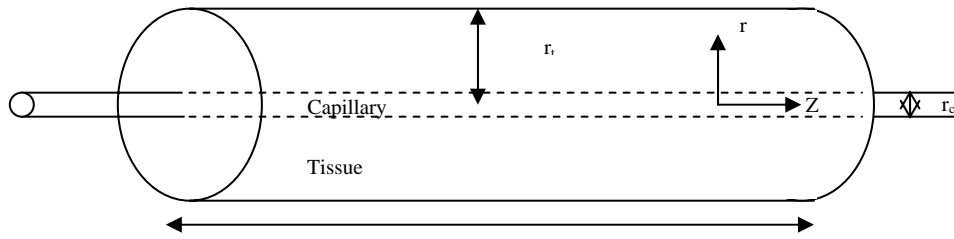


Figure 1. Geometry of Mathematical Model used.

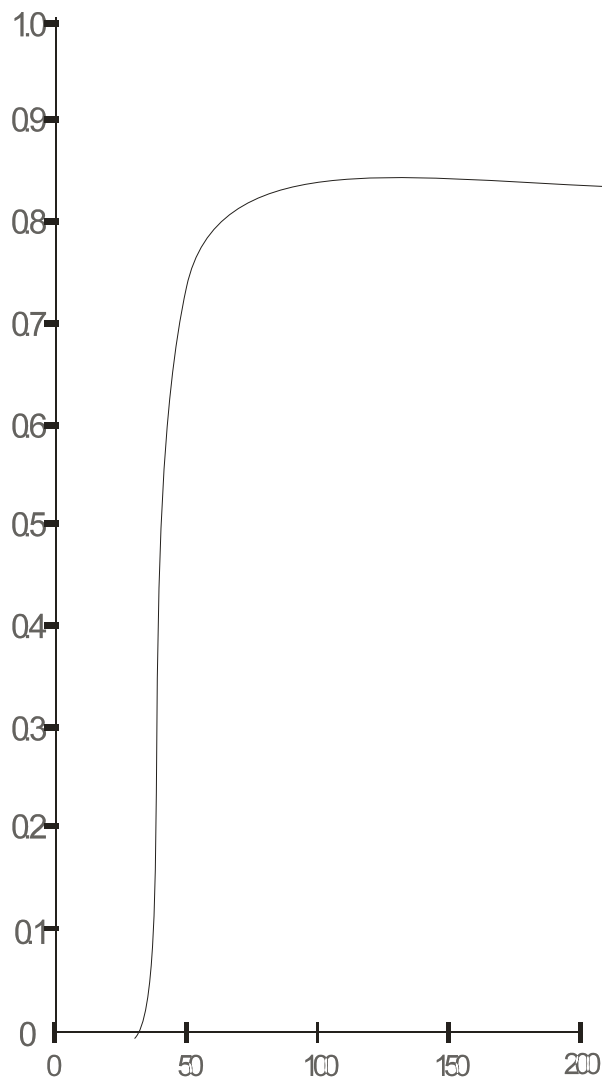


Figure 2. The oxyhaemoglobin dissociation curve

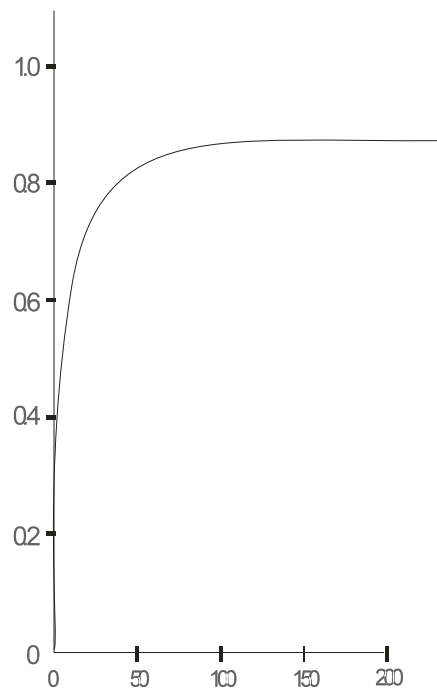


Figure 3. The oxyhaemoglobin dissociation curve

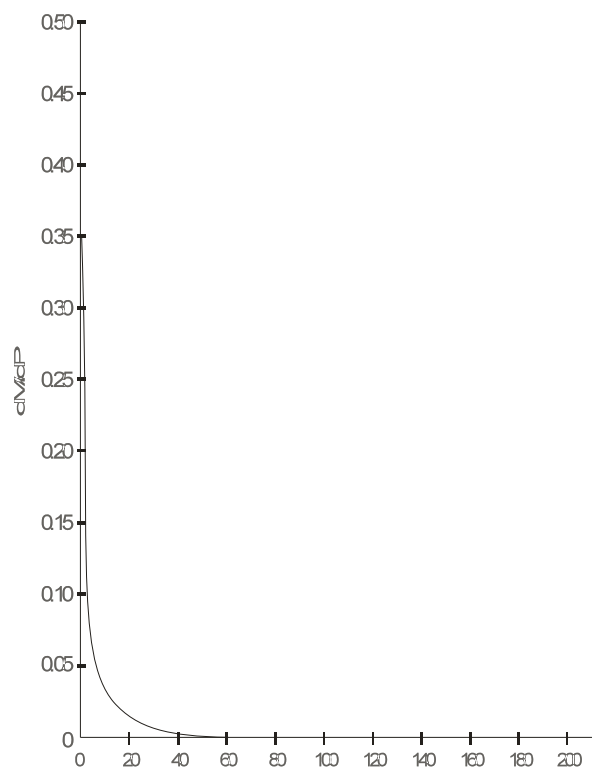


Figure 4. The derivative of the myoglobin dissociation curve

A_t the reaction term, (after combining both the equations (7) and (8)), can be written as

$$A_t = C_m u_m \frac{M - g(p)}{1 - g(p)} \quad (9)$$

Boundary conditions for both the set of equations are given below:

(i) No diffusion flux of Myoglobin outside the tissue

$$\frac{\partial M}{\partial r} = 0, \quad r = r_c, r_t \quad 0 \leq z \leq b$$

$$\frac{\partial M}{\partial z} = 0, \quad z = 0, b \quad r_c \leq r \leq r_t$$

(ii) No diffusion flux of hemoglobin outside capillary across any other boundary

$$\frac{\partial H}{\partial r} = 0, \quad r = r_c, \quad 0 \leq z \leq b$$

$$\frac{\partial H}{\partial z} = 0, \quad z = a, \quad 0 \leq r \leq r_c$$

(iii) p and the flux of oxygen are continuous across the capillary tissue interface, therefore the condition are:

$$p \Big|_{z=r_c^-} = p \Big|_{z=r_c^+} \quad 0 \leq z \leq b$$

$$a_p D_p \frac{\partial p}{\partial r} \Big|_{z=r_c^-} = a_t D_t \frac{\partial p}{\partial r} \Big|_{z=r_c^+} \quad 0 \leq z \leq b$$

(iv) Hemoglobin saturation is in equilibrium with partial pressure of oxygen in blood entering the capillary

$$H = f(p_a) ; z = 0 \quad 0 \leq r \leq r_c$$

(v) The flux-dissolved oxygen per unit area is given by $D \nabla$, and is zero across closed boundaries. In addition to that we assume a non-diffusion flux condition where blood flows out of the capillary and so

$$\frac{\partial p}{\partial z} = 0 \quad z = 0, \quad r_c \leq r \leq r_t$$

$$\frac{\partial p}{\partial r} = 0 \quad r = r_t, \quad 0 \leq z \leq b$$

$$\frac{\partial p}{\partial z} = 0 \quad z = b, \quad 0 \leq r \leq r_t$$

(vi) Radial symmetry:

$$\frac{\partial p}{\partial r} = \frac{\partial H}{\partial r} = 0 \quad \text{at } r = 0, \quad 0 \leq z \leq b$$

(vii) P is initially in equilibrium with arterial blood:

$$p = p_a ; z = 0, \quad 0 \leq r \leq r_c$$

(viii) For a unique solution of fractional myoglobin saturation M , we have:

$$M(b, r_t) = g(p(b, r_t))$$

In the next step we non-dimensionalize the model equations, where we choose scaling such that the non-dimensional variables range from 0 to 1. Here L is the length scale to be chosen appropriately.

$$\left. \begin{aligned} p &\rightarrow p_0 + (p_1 - p_0)p \\ H &\rightarrow H_0 + (H_1 - H_0)H \\ r &\rightarrow r_c r \\ z &\rightarrow L_z z \end{aligned} \right\} \quad (10)$$

then equation (1) and (2) turn to the equations

$$\begin{aligned} A_1 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) + A_2 \frac{\partial^2 p}{\partial z^2} \\ = \frac{\partial p}{\partial z} + A_3 \frac{f(p)}{(1 - H_0)/(H_1 H_0) + f(p)} \end{aligned} \quad (11)$$

$$\begin{aligned} B_1 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial H}{\partial r} \right) + B_2 \frac{\partial^2 H}{\partial z^2} \\ = \frac{\partial H}{\partial z} - B_3 \frac{f(p)}{(1 - H_0)/(H_1 H_0) + f(p)} \end{aligned} \quad (12)$$

Neglecting the terms of $0(10^{-2})$ the size of the largest terms,

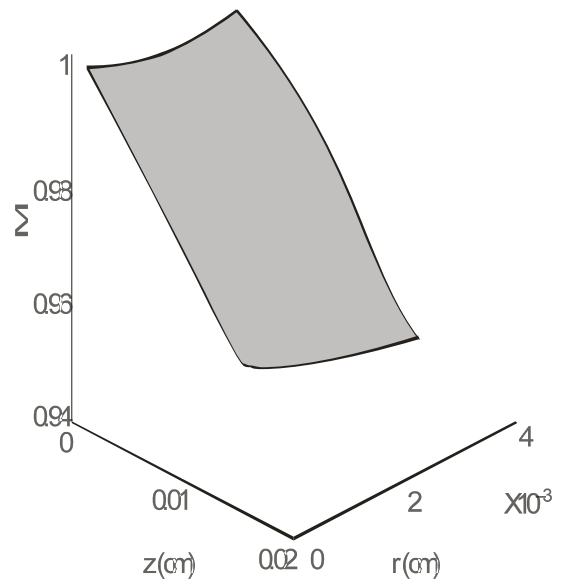
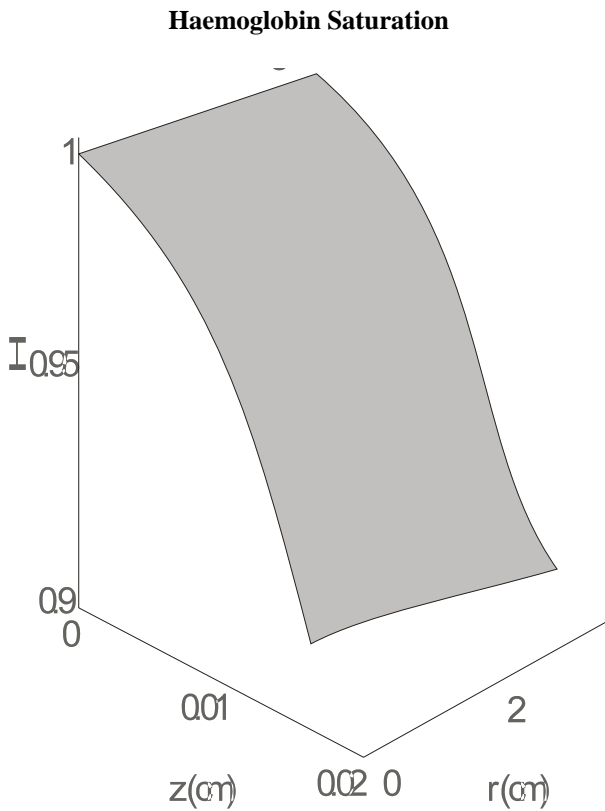
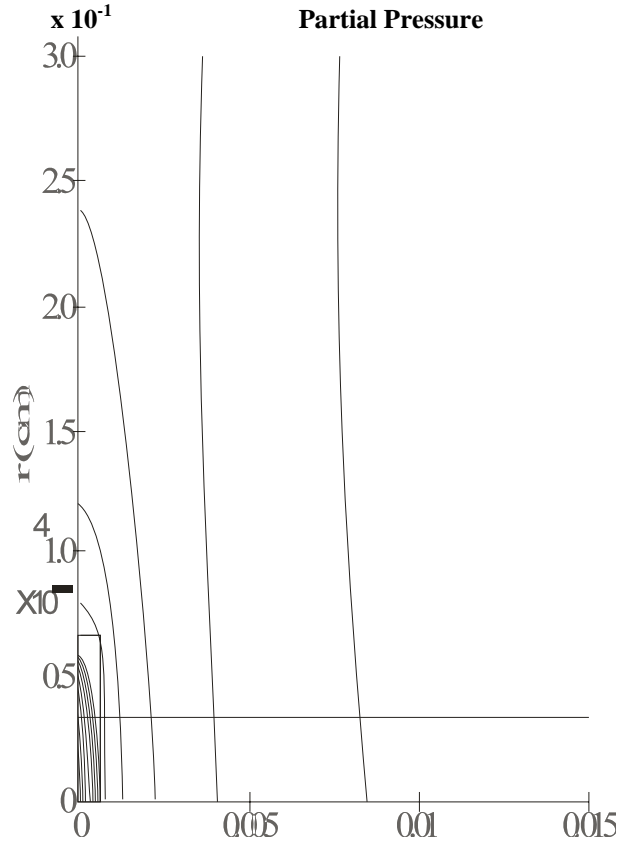
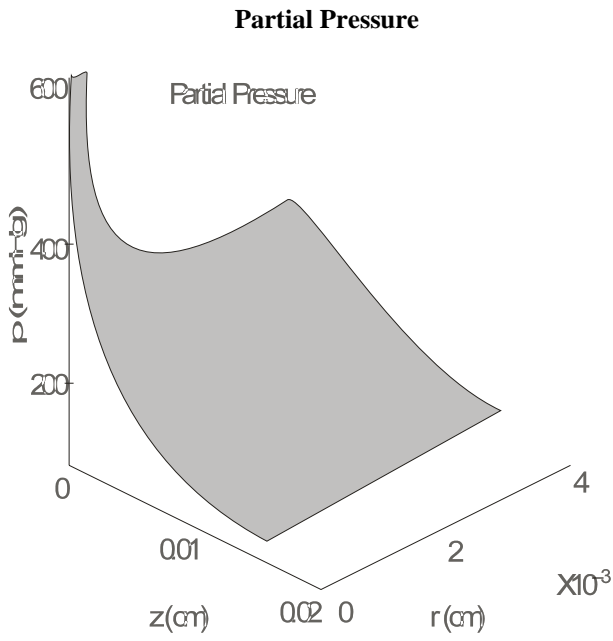


Figure 5. The solution of the governing equations, Eqs. 1-6 and 9 with $P_a = 600$ mmHg.

$$A_1 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) + A_2 \frac{\partial^2 p}{\partial z^2} = \frac{\partial p}{\partial z} + A_3 \frac{f(p)}{1+f(p)} \quad (13)$$

$$0 = \frac{\partial H}{\partial z} - B_3 \frac{f(p)}{1+f(p)} \quad (14)$$

With the help of (13) and (14) we have

$$A_1 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) + A_2 \frac{\partial^2 p}{\partial z^2} = \left(1 + \frac{A_3}{B_3} f'(p) \right) \frac{\partial p}{\partial z} \quad (15)$$

3. TISSUE REGION

Using the parameters

$$p \rightarrow p_0 + (p_1 - p_0)p,$$

$$M \rightarrow M_0 + (m_1 - m_0)M$$

$$r \rightarrow L_r r \quad z \rightarrow L_z z \quad (16)$$

equation (5) and (6) turn to

$$\begin{aligned} & \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) + \frac{L_r^2}{L_z^2} \frac{\partial^2 p}{\partial z^2} \\ &= -C_1 \frac{g(p)}{\frac{1-M_0}{M_1-M_0} + g(p)} + C_2 \end{aligned} \quad (17)$$

$$\begin{aligned} & \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial M}{\partial r} \right) + \frac{L_r^2}{L_z^2} \frac{\partial^2 M}{\partial z^2} \\ &= -D_1 \frac{g(p)}{\frac{1-M_0}{M_1-M_0} + g(p)} + D_2 \end{aligned} \quad (18)$$

On neglecting terms of size $O(10^{-2})$, the magnitude from equation (18) is given by

$$D_1 \frac{g(p)}{1+g(p)} = 0 \quad (19)$$

from which we deduce that we can assume an instantaneous reaction and $g(p) = 0$.

4. RESULTS AND DISCUSSION

For the purpose of our numerical calculation we use the following values different terms used in above model

$$r_c = 3.3 \times 10^{-4} \text{ cm} \quad r_t = 3.3 \times 10^{-3} \text{ cm} \quad b = 1.6 \times 10^2 \text{ cm}$$

$$v = 0.02 \text{ unit} \quad D_p = 1.0 \times 10^{-5} \text{ unit} \quad D_t = 1.6 \times 10^{-5} \text{ unit}$$

$$D_h = 1.4 \times 10^{-7} \text{ unit} \quad a_p = 1.53 \times 10^{-9} \text{ unit}$$

$$D_m = 5.0 \times 10^{-7} \text{ unit} \quad C_h = 8.9 \times 10^{-6} \text{ unit} \quad C_m = 2.7 \times 10^{-7} \text{ unit}$$

$$U_h = 60 \text{ unit}$$

$$U_h' = 2 \times 10^{10} \text{ unit} \quad Q = 4.5 \times 10^{-8} \text{ unit} \quad a_t = 1.3 \times 10^{-9} \text{ unit}$$

5. AXIAL DIFFUSION

On the basis of our results, we come to know that the axial diffusion term should not be neglected as many workers have neglected it in their solution of the governing equations. In tissue region, axial diffusion is nearly 15 times smaller than the radial diffusion and so should be avoided by equation (17) and (18). We can show the importance of axial diffusion. As the ratio $(L_r / L_z) \in [0,1]$ increases axial diffusion term becomes large. Salathe et al [5] have also investigated boundary layers in region of the inflow boundary using matched asymptotic expansion.

6. MYOGLOBIN

On the basis of our results, we observed that on increasing consumption of oxygen, p is lowered; therefore we cannot neglect myoglobin because myoglobin plays an emergency storehouse of oxygen in case of deficiency of oxygen.

7. BOUNDARY CONDITIONS

We have come to know that the transport is convection dominated in the axial direction inside the capillary. This will be a small order diffusive flux of oxygen along the inflow tube and so the assumption $p - p_a$ for blood inflow geometry based is valid. In case of ant flew geometry we can assume diffusion is of neglecting order.

8. SYMBOLS USED IN MODEL

R	Radial co-ordinate
Δ_p	Diffusion coefficient of oxygen in plasma
p	Oxygen partial pressure
H	Fractional hemoglobin saturation
Q_p	Oxygen solubility in plasma
Δ_t	Diffusion coefficient of oxygen in tissue
D_p	Diffusion coefficient of hemoglobin
D_m	Diffusion coefficient of myoglobin
C_h	Oxygen carrying capacity of hemoglobin
r_c	Capillary radius
C_m	Oxygen carrying capacity of myoglobin
b	Length of capillary
U_h	Backward reaction rate for oxy-hemoglobin reaction.
r_t	Tissue radius
U_m	Backward reaction rate for oxy-myoglobin reaction
Q_t	Oxygen solubility in tissue

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