A NEW APPROACH FOR VIBRATION ANALYSIS OF A CRACKED BEAM

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Abstract In this paper the equations of motion and corresponding boundary conditions for bending vibration of a beam with an open edge crack has been developed by implementing the Hamilton principle. A uniform Euler-Bernoulli beam has been used in this research. The natural frequencies of this beam have been calculated using the new developed model in conjunction with the Galerkin projection method. The crack has been modeled as a continuous disturbance function in displacement field which could be obtained from fracture mechanics. The results show that the natural frequencies of a cracked beam reduce by increasing crack depth. There is an excellent agreement between the theoretically calculated natural frequencies and those obtained using the finite element method.

Keywords: Cracked beam, Continuous model, natural frequency

چکیده در این مقاله معادلات دیفرانسیل حرکت و شرایط مرزی متناظر با آن برای ارتعاشات خمشی یک تیر که دارای یک ترک باز لبه‌ایست با بکارگیری اصل همیلتون استخراج گردیده است. تیر مورد نظر در این پژوهش یک تیر اویلر-برنولی یکنواخت فرض شده است. همچنین فرکانسهای طبیعی متناظر با مدل استخراج شده در این مقاله با استفاده از روش گالرکین بدست آمده است. ترک بصورت یک اغتشاش پیوسته در میدان جابجایی تیر مدل شده است که این تابع اغتشاش را می‌توان با استفاده از مکانیک شکست تعیین نمود. نتایج بدست آمده نشان می‌دهند که فرکانسهای طبیعی تیر با افزایش عمق ترک کاهش می‌یابند. تطابق بسیار خوبی نیز میان فرکانسهای طبیعی محاسبه شده از تئوری توسعه یافته در این مقاله و فرکانسهای طبیعی بدست آمده از روش اجزاء محدود وجود دارد.

1. INTRODUCTION

Dynamic structures subjected to periodic loads compose a very important part of industrial machineries. One of the major problems in these machineries is the fatigue and the cracks initiated by the fatigue. These cracks are the most important cause of accidents and failures in industrial machinery. In addition, existing of the cracks may cause vibration in the system. Thus an accurate and comprehensive investigation about vibration of cracked dynamic structures seems to be necessary. On the base of these investigations the cracks can be identified well in advance and appropriate measures can be taken to prevent more damage to the system due to the high vibration level.

The dynamic behavior of cracked structures has been investigated by many researchers.

Dimaragonas (1996) presented a review on the researches have been done on the issue of vibration of cracked structures until 1996 [1]. His review contains vibration of cracked rotors, bars, beams, plates, pipes, blades and shells. Among all these structural elements, the rotors have been focused wider because of their applications. Two literature reviews are also available on the dynamic behavior of cracked rotors by Wauer (1990) and Gasch (1993) [2, 3].

Cracked beam is also one of the structural elements which has been studied by researchers. For the fist time, Dimaragonas (1983) suggested an analytical method for the computation of dynamic response of a cracked Euler-Bernoulli beam by modeling the cracked region as a local flexibility resulted from fracture mechanics [4]. This local flexibility idea has been followed by several researchers till now [5,6]. In this research the crack is supposed to be a rotational spring between two continuous parts of the beam.

Christides and Barr (1986) developed a continuous theory for vibration of a uniform Euler-Bernoulli beam containing one or more pairs of symmetric cracks. A differential equation of motion and corresponding boundary conditions are given in this paper using the Hu-Washizu variational principle [7]. They assumed that the effect of the crack can be taken into account by modification in the stress field. But in this model different and incompatible assumptions has been made for displacement field and strain field. Shen and Pierre (1990) presented a similar model for bending vibration of a cracked beam with symmetric cracks [8]. They used a two dimensional finite element method to obtain parameters related to the stress concentration profile near the tip of the crack. They have (1994) developed also a continuous model for bending vibration of a cracked Euler-Bernoulli beam with a single edge crack by implementation of the Hu-Washizu variational theory [9]. Similar to Christides and Barr research, the displacement and strain fields has been chosen independently and thus these fields are not compatible. Furthermore this model is very similar to their previous model except the stress disturbance function due to the crack has some modifications. This function depends on some constants which has been calculated from the finite element results. Carneiro and Inman (2001) suggested some modifications for the Shen and Pierre model [10]. They discussed that the differential equation presented by Shen and Pierre is not self-adjoint and thus can not have real eigen values.

Chondros et al.(1998) have developed a continuous model, differential equation of motion and the appropriate boundary conditions for lateral vibration of a one dimensional beam with rectangular cross section using Hu-Washizu-Barr formulation [11]. In their following paper (2001) they have extended the work and obtained the first natural frequency for a rectangular cross-section uniform beam containing a breathing crack [12]. In their model, the crack was supposed to be a continuous flexibility and its effect was modeled as a disturbance in the displacement field of the beam. The crack disturbance function was obtained using the strain energy density at the cracked area. Their model may not be applicable for general cases because of some special assumptions. The incompatibility between displacement and strain fields may also be seen in this work.

Yang et al. (2001) have presented an energy based numerical model to investigate the influence of cracks on structural dynamic characteristics during the vibration of a beam with open cracks [13]. The using model is originally same as Christides and Barr model.

Zheng and Fan (2003) developed a simple tool for the vibration and stability analysis of cracked hollow-sectional beams. The local flexibility approach has been used in this work [14].

Lin (2003) presented an analytical transfer matrix method to solve the direct and inverse problems of simply supported beams with an open crack. The crack is modeled as a rotational spring with sectional flexibility. By using the Timoshenko beam theory on two separate beams respectively and applying the compatibility requirements of the crack, the characteristic equation for this cracked system has been obtained explicitly [15].

Zheng and Kessissoglou (2004) calculated the natural frequencies and mode shapes of a cracked beam using the finite element method. An ‘overall additional flexibility matrix’, instead of the ‘local additional flexibility matrix’, is added to the flexibility matrix of the corresponding intact beam element to obtain the total flexibility matrix, and therefore the stiffness matrix [16].

The present study is a new approach toward the dynamical behavior of cracked beams. In this paper a modification in the displacement field of the beam similar to the Shen and Pierre model has been suggested but the strain field is not assumed independently. Instead, the strain field is calculated directly from displacement field. Furthermore the corresponding constants have been evaluated from the fracture mechanics. The equation of motion has been obtained from the variation theory and the Hamilton Principle. This equation has less complexity compared to other models. The validity of the obtained results has been confirmed by comparison with the finite element results.

2. EQUATION OF MOTION

A prismatic uniform Euler-Bernoulli beam with an open edge crack which is simply supported and subjected only to pure bending is shown in figure (1).

In fig. (1) a, d, L, and x_c denote the crack depth, half depth of the beam, beam length and the position of the crack respectively. If the beam does not contain any crack, the Euler-Bernoulli beam theory suggests:

$$u(x,z,t) = -z\frac{\partial w(x,t)}{\partial x} \qquad (1)$$

In which u(x,z,t) is displacement in the x-direction and w(x,t) is the vertical displacement of the neutral axis. The normal strain in this beam can be obtained as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \qquad (2)$$

For an Euler-Bernoulli beam with an open edge crack one has [9]:

$$\frac{\partial u(x,z,t)}{\partial x} = (-z + \varphi(x,z))\frac{\partial^2 w}{\partial x^2} \qquad (3)$$

In this equation the function $\varphi(x,z)$ is called the crack disturbance function. This function will be obtained later in this paper. Now the normal strain for this beam can be calculated as follows:

$$\varepsilon_{xx} = \frac{\partial u(x,z,t)}{\partial x} = (-z + \varphi(x,z))\frac{\partial^2 w}{\partial x^2} \qquad (4)$$

This strain is assumed to be the only non-zero strain value in the xz-plane. By implementing the Hook's law it can be written as:

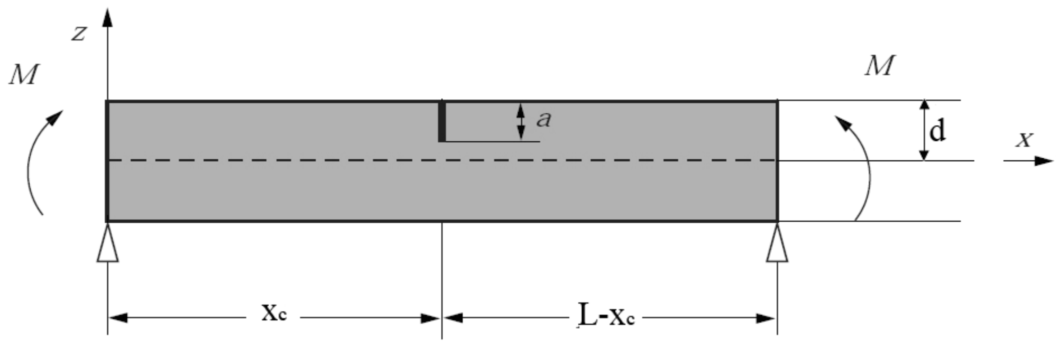

Figure 1. Geometry of a simply support Euler-Bernoulli beam with an open edge crack

$$\sigma_{xx} = E\varepsilon_{xx} = E(-z+\varphi(x,z))\frac{\partial^2 w}{\partial x^2} \quad (5)$$

In which, E is the modulus of elasticity. Thus the strain energy will be:

$$V = \iiint \frac{1}{2}\sigma_{xx}\varepsilon_{xx}dxdydz \quad (6)$$

Substituting from equations (4) and (5):

$$V = \frac{1}{2}E\iiint (-z+\varphi(x,z))^2(\frac{\partial^2 w}{\partial x^2})^2 dxdydz \quad (7)$$

The kinetic energy is also as follows:

$$T = \iiint \frac{1}{2}\rho(\frac{\partial w}{\partial t})^2 dxdydz \quad (8)$$

Due to the Euler-Bernoulli beam theory the rotational moment of inertia has been neglected. Using Lagrangian equation one has:

$$L = T - V = \frac{1}{2}\iiint [\rho(\frac{\partial w}{\partial t})^2 - E(-z+\varphi(x,z))^2(\frac{\partial^2 w}{\partial x^2})^2]dxdydz \quad (9)$$

If the beam is assumed to be a prismatic beam one can conclude that:

$$L = \frac{1}{2}\int_0^L [\rho A(\frac{\partial w}{\partial t})^2 - Ef(x)(\frac{\partial^2 w}{\partial x^2})^2]dx \quad (10)$$

In this equation A is the cross-sectional area of the beam and f(x) can be given by:

$$f(x) = \iint (-z+\varphi(x,z))^2 dydz \quad (11)$$

From the Hamiltonian principle:

$$\delta\int_{t_0}^{t_1} Ldt = 0 \quad (12)$$

By substituting equation (10) into (12) one can write:

$$\int_{t_0}^{t_1}\int_0^L [\rho A(\frac{\partial w}{\partial t})\delta(\frac{\partial w}{\partial t}) - Ef(x)(\frac{\partial^2 w}{\partial x^2})\delta(\frac{\partial^2 w}{\partial x^2})]dxdt = 0 \quad (13)$$

After integrating by parts equation (13) can be written as:

$$f(x)\frac{\partial^2 w}{\partial x^2}\delta\frac{\partial w}{\partial x}\Big|_0^L - \frac{\partial}{\partial x}(f(x)\frac{\partial^2 w}{\partial x^2})\partial w\Big|_0^L + \int_{t_0}^{t_1}\int_0^L\left[\rho A(\frac{\partial^2 w}{\partial x^2}) + E\frac{\partial^2}{\partial x^2}(f(x)\frac{\partial^2 w}{\partial x^2})\right]\delta w dxdt = 0 \quad (14)$$

Thus the field equation of motion for the cracked Euler-Bernoulli beam is:

$$E\frac{\partial^2}{\partial x^2}(f(x)\frac{\partial^2 w}{\partial x^2}) + \rho A(\frac{\partial^2 w}{\partial t^2}) = 0 \quad (15)$$

And the Boundary conditions are:

$$\begin{cases} f(x)\frac{\partial^2 w}{\partial x^2} = 0 & or \quad \delta\frac{\partial w}{\partial x} = 0 & at \quad x = 0 \\ f(x)\frac{\partial^2 w}{\partial x^2} = 0 & or \quad \delta\frac{\partial w}{\partial x} = 0 & at \quad x = L \\ \frac{\partial}{\partial x}(f(x)\frac{\partial^2 w}{\partial x^2}) = 0 & or \quad \delta w = 0 & at \quad x = 0 \\ \frac{\partial}{\partial x}(f(x)\frac{\partial^2 w}{\partial x^2}) = 0 & or \quad \delta w = 0 & at \quad x = L \end{cases} \quad (16)$$

It is obvious that for an uncracked beam the function f(x) will be equal to I (the moment of inertia) and the governing equation of motion (15) and its corresponding boundary conditions (16) will turn into the ordinary Euler-Bernoulli beam equation and boundary conditions.

3. CRACK DISTURBANCE FUNCTION

The stress and strain distribution in elastic bodies has been studied by Irwin [17] and Paris and Shin

[18]. They have shown that the normal stress σ_{xx} concentrates at the crack tip and decays in inverse proportion to the square root of the distance from the crack tip. This phenomenon is reproduced here using the crack disturbance function $\varphi(x,z)$. This function must be maximum at the crack tip. It is also taken to decay exponentially along the length of the beam and to vary linearly through the depth of uncracked portion of beam[9]:

$$\varphi(x,z)=\left[z-m(z+\frac{a}{2})u(d-a-z)\right]e^{-\frac{\alpha}{d}|x-x_c|} \tag{17}$$

In equation (17), x_c, a and d present the crack position, the crack length and the half depth of the beam as shown in figure 1. The positive non-dimensional constant α determines the rate of stress decay from the crack tip. The constant m represents the slope of the linear stress distribution at the cracked section. $u(d-a-z)$ is the unit step function and can be written as follows:

$$u(d-a-z)=\begin{cases}1 & z<d-a\\ 0 & z\geq d-a\end{cases} \tag{18}$$

Here, the two constants α and m are unknown. Before solving the equation (15) for natural frequencies, these two parameters of the crack disturbance function is required to be determined.

It is assumed that the beam is loaded only with a pure bending moment M. Under general loading conditions the additional displacements u^*, w^*, θ^* will initiate due to the crack. If the beam was subjected only to pure bending M, the additional rotation θ^* can be obtained from Castigliano's theorem as follows [19]:

$$\theta^*=\frac{\partial U_T}{\partial M} \tag{19}$$

Where, U_T is the strain energy due to the crack. This strain energy has the following form:

$$U_T=\int_{CRACK} J(\alpha)\,d\alpha \tag{20}$$

This equation is called Paries' Equation. In fact, the integral in equation (20) is an integral over a surface as follows [15]:

$$U_T=\int_{-b}^{b}\int_{0}^{a} J_s(\alpha)\,d\alpha\ dy \tag{21}$$

In which, a is the crack depth and J_s is the strain energy density which could be obtained from the following equation:

$$J_s=\frac{1}{E'}\left[\left(\sum_{i=1}^{6}K_{Ii}\right)^2+\left(\sum_{i=1}^{6}K_{IIi}\right)^2+\left(\sum_{i=1}^{6}K_{IIIi}\right)^2\right] \tag{22}$$

In this equation, if the plane stress assumption was used then $E=E'$ and if the plane strain assumption is used then $E'=\frac{E}{1-\nu}$. In this article the plain strain assumption is used. In addition, in equation (14), $m=1+\nu$ and $K_{Ii}, K_{IIi}, K_{IIIi}$ are the Stress Intensity Factors (SIF) corresponding to three fracture modes and for ith loading. In fracture mechanics, the values SIF are well known for a strip of unit thickness with a transverse crack [13]. The stress intensity factor for a single edge cracked beam under pure bending is:

$$K_I=\sigma_0\sqrt{\pi a}\,F_I(\frac{a}{2d}) \tag{23}$$

Where:

$$\sigma_0=\frac{Md}{I} \tag{24}$$

And

$$F_I(\alpha)=1.12-1.4\alpha+7.33(\alpha)^2-13.1(\alpha)^3+14(\alpha)^4 \tag{25}$$

Which has an accuracy of $\pm 0.2\%$ for $\frac{a}{2d}<0.6$ that is an appropriate accuracy for engineering purposes. Since the beam is only subjected to bending moment M, the stress intensity factors in modes II and III of fracture are zero. Thus it can be written:

$$J_s = \frac{K_I^2}{E'} = \frac{1-\nu^2}{E}\sigma_0^2 \pi a F_1^2 (\frac{a}{2d}) \quad (26)$$

By substituting equation (26) into equation (21) and integrating over the crack surface, the amount of additional rotation θ^* from equation (19) can be obtained as:

$$\theta^* = \Phi(\frac{a}{2d})\frac{M}{EI} \quad (27)$$

In which:

$$\Phi_I(\alpha) = 3\pi d(1-\nu^2)(0.63\alpha^2 - 1.045\alpha^4 - 9.97\alpha^5 + 20.29\alpha^6 - 33.03\alpha^7 + 47.11\alpha^8 - 40.76\alpha^9 + 19.6\alpha^{10}) \quad (28)$$

On the other hand, for a healthy simply supported beam subjected to a constant moment M at two ends one has:

$$\frac{d^2 w_h}{dx^2} = \frac{M}{EI} \quad (29)$$

By integrating from equation (29) and substituting the corresponding boundary conditions, one has:

$$\theta_h(x) = \frac{dw_h}{dx} = \frac{M}{2EI}(2x - l) \quad (30)$$

And for beams with a single edge crack from the previous section one can write:

$$\frac{d^2 w_c}{dx^2} = \frac{M}{Ef(x)} \quad (31)$$

By integrating from this equation and substituting the corresponding boundary conditions, one has:

$$\theta_c(x) = \frac{dw_c}{dx} = \frac{M}{E}B(x) - \frac{M}{El}\int_0^L B(x)dx \quad (32)$$

In which:

$$B(x) = \int \frac{1}{f(x)}dx \quad (33)$$

Now, it can be cited that:

$$\begin{cases} \theta_c(0) - \theta_h(0) = \theta^* \\ \theta_c(L) - \theta_h(L) = -\theta^* \end{cases} \quad (34)$$

In the system of nonlinear equations (34) the only unknown parameters are α and m which can be obtained from the numerical solution of these equations for several crack depth ratios $\frac{a}{2d}$.

After evaluating the parameters α and m, the crack disturbance function will define completely and the governing equation of motion (15) can be solved for natural frequencies.

4. NATURAL FREQUENCIES

As shown in figure 1, a beam with an open edge crack and subjected to pure bending at two ends is considered. The equation of motion of this beam is given in equation (15). This equation is a self-adjoint equation and thus it has the real eigen values. However the eigen value problem for this equation can not be solved explicitly for the natural frequencies because of the variable coefficient f(x). Thus an approximation method must be used to evaluate the natural frequencies. In this paper the Galerkin projection method has been used for this purpose.

By separating the variable $w(x,t)$ in equation (15) one has:

$$w(x,t) = X(x)T(t) \quad (35)$$

Thus the eigenvalue problem for a simply supported beam will be as follows:

$$\begin{cases} \frac{d^2}{dx^2}(Ef(x)\frac{d^2X(x)}{dx^2}) = \lambda\rho AX(x) \\ X(0) = X(L) = 0 \\ X''(0) = X''(L) = 0 \end{cases} \quad (36)$$

If a periodic solution does exist then $\omega_n = \lambda_n^{\frac{1}{2}}$ are the natural frequencies. In the Galerkin method the mode shape function $X(x)$ is assumed to be a linear combination of some trial or shape functions as follows:

$$X(x) = \sum_{i=1}^{N} c_i S_i(x) \tag{37}$$

Each shape function must at least satisfy the physical boundary conditions of the problem. The appropriate shape functions in this case seem to be as:

$$S_i(x) = \sin\frac{i\pi x}{L} \tag{38}$$

Now, for the steady state periodic response the function *T(t)* can be written as follows:

$$T(t) = e^{i\omega t} \tag{39}$$

Substituting equation (39) and (37) into equation (15) and multiplying the two sides of this equation into $S_m(x)$ and integrating over the length of the beam, the eigenvalue problem will turn into the following form:

$$\det(K - \omega^2 M) = 0 \tag{40}$$

In which K is a $N \times N$ stiffness matrix with the following elements:

$$k_{ij} = \int_0^L \frac{d^2 S_i(x)}{dx^2} Ef(x) \frac{d^2 S_j(x)}{dx^2} dx \tag{41}$$

And M is a $N \times N$ mass matrix with the following elements:

$$m_{ij} = \int_0^L S_i(x) \rho A S_j(x) dx \tag{42}$$

To verify the convergence of Galerkin's method, several numbers of trial functions has been tested in the calculations. Each time the percentage of error relative to the previous step has been obtained. The results indicate with increasing the value of N the approximation accuracy will increase. In this paper the value of N is set to be 100 for a reasonable approximation.

5. FINITE ELEMENT ANALYSIS

In this investigation a finite element analysis for a rectangular beam with an open edge crack also has been performed to verify the analytical results obtained in previous section. In this analysis a high density mesh is used.

The most important region in a fracture model is the region around the edge of the crack. We will refer to the edge of the crack as a *crack tip* in a 2-D model and *crack front* in a 3-D model. This is illustrated in figure (2).

As mentioned above, in linear elastic problems, it has been shown that the displacements near the crack tip (or crack front) vary as a function of $\sqrt{r}$, where r is the distance from the crack tip. The stresses and strains are singular at the crack tip, varying as a function of $\frac{1}{\sqrt{r}}$. To pick up the singularity in the strain, the crack faces should be coincident, and the elements around the crack tip (or crack front) should be quadratic, with the midside nodes placed at the quarter points. Such elements are called *singular elements*. Figure (3) shows such elements in 2D and 3D.

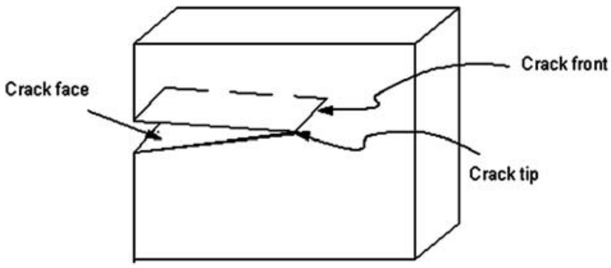

Figure 2. Region around the tip of the crack

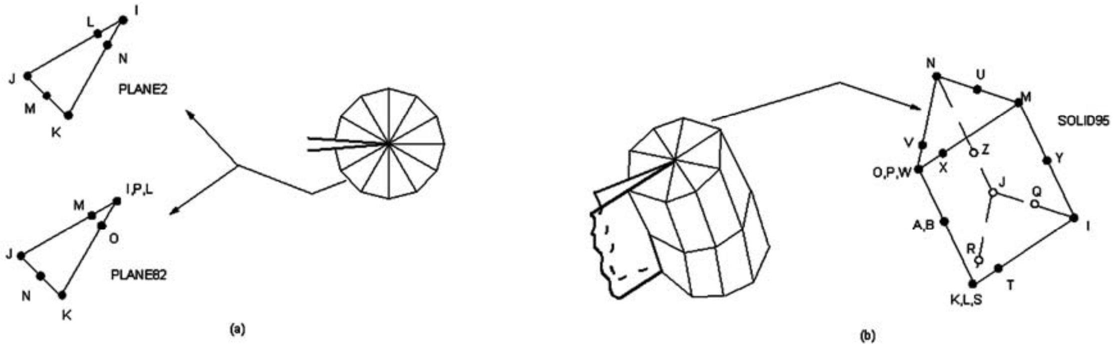

Figure 3. Singular elements in 2D and 3D [20]

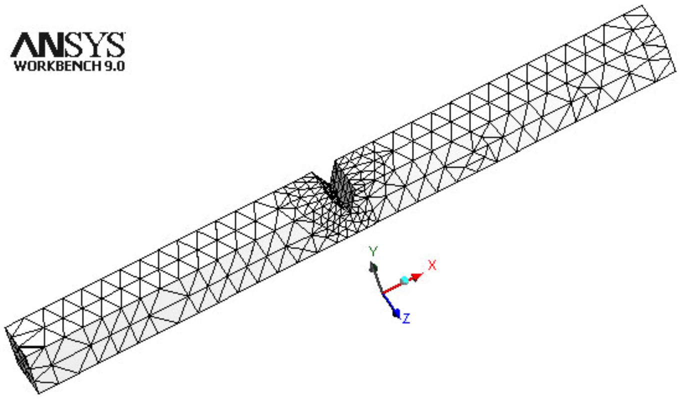

Figure 4. A cracked beam after meshing process

The ANSYS software has been used for the finite element analysis. In this analysis the singular elements are used for meshing process and the natural frequencies have been obtained. Figure (4) shows a cracked beam model after meshing process.

6. RESULTS AND DISCUSSION

The first four natural frequencies of a cracked simply supported beam has been evaluated in this paper. It is assumed that the beam has been made of steel and has a rectangular cross-section and the length of the beam is 24.65 times greater than its depth ($\frac{L}{2d} = 24.65$). In the basis of the developed theory in section 2 and the Galerkin method described in section 4, the first four natural frequencies of a cracked beam with several crack-depth ratio and several positions of crack has been obtained. The finite element results are also evaluated for these beams.

Figure (5) shows the first natural frequency of a cracked beam divided to the corresponding natural frequency of an uncracked beam versus the crack depth ratio. As it can be seen in this figure the first natural frequency will reduce when the crack depth ratio increases. Regarding this figure the finite element results have a good agreement with the analytical results.

Figure (6) shows the second natural frequency for a crack beam with several crack poison ratios. It can be seen that the natural frequency reduction in the second mode is lass than the first mode. Figures (6) and (7) show the third and forth natural frequency ratio for the mentioned beam respectively.

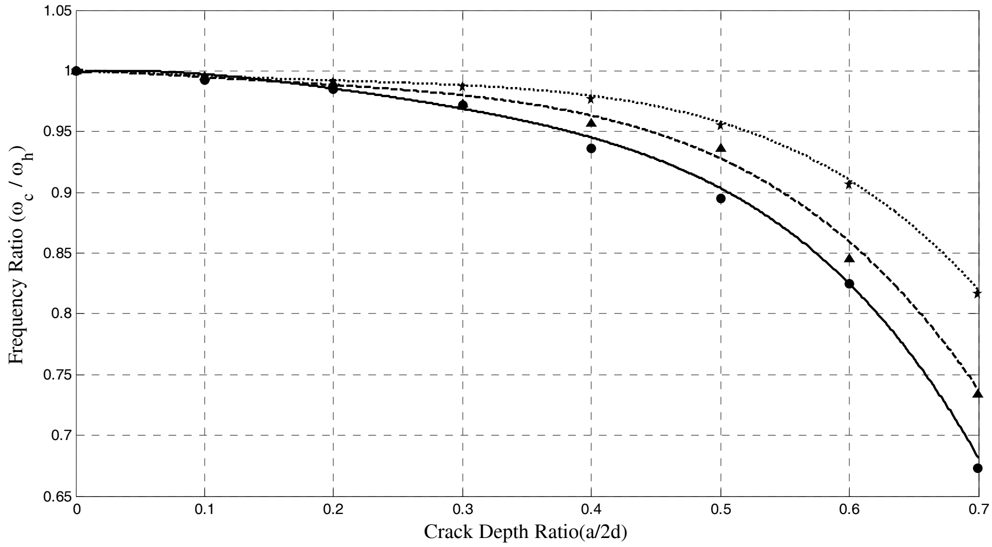

Figure 5. First natural frequency ratio.

$x_C/L = 0.2$:analytical results(___) , finite element results(•); $x_C/L = 0.3$:analytical results(----) , finite element results(▲); $x_C/L = 0.4$:analytical results(….) , finite element results(★);

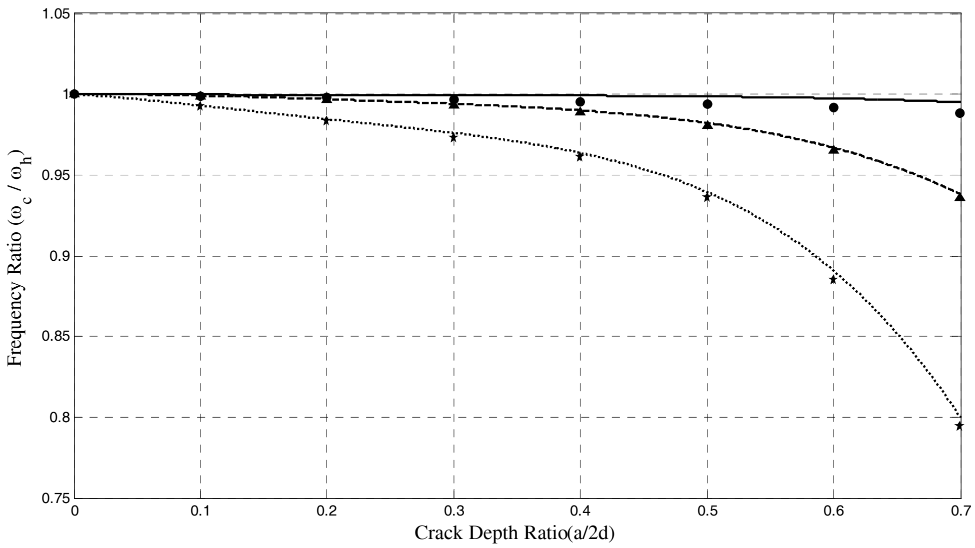

Figure 6. Second natural frequency ratio.

$x_C/L = 0.2$:analytical results(___) , finite element results(•); $x_C/L = 0.3$:analytical results(----) , finite element results(▲); $x_C/L = 0.4$:analytical results(….) , finite element results(★);

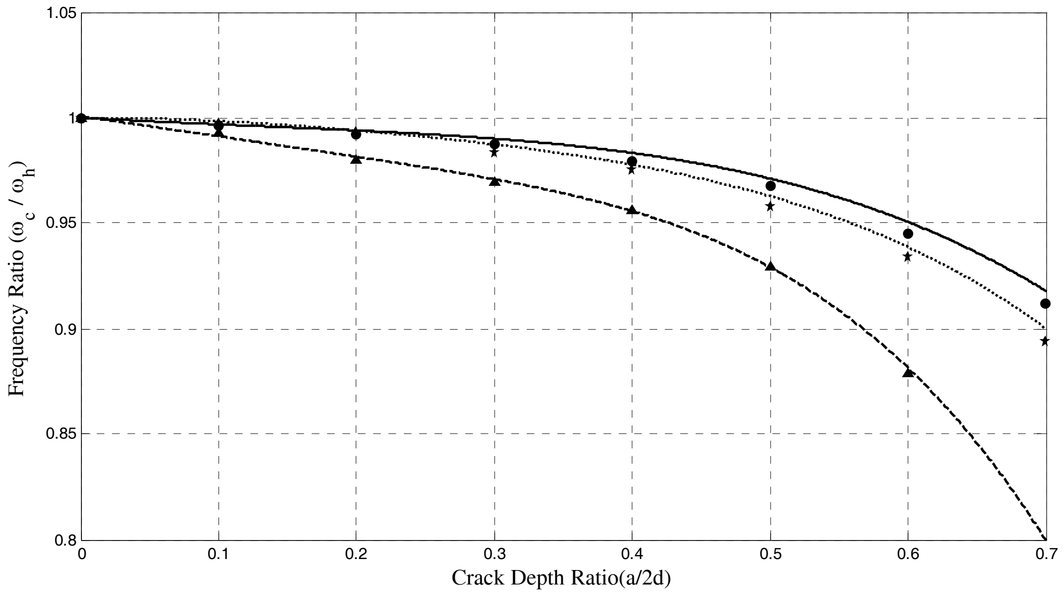

Figure 7. Third natural frequency ratio.

$x_C/L = 0.2$:analytical results(___) , finite element results(•); $x_C/L = 0.3$:analytical results(----) , finite element results(▲); $x_C/L = 0.4$:analytical results(….) , finite element results(★);

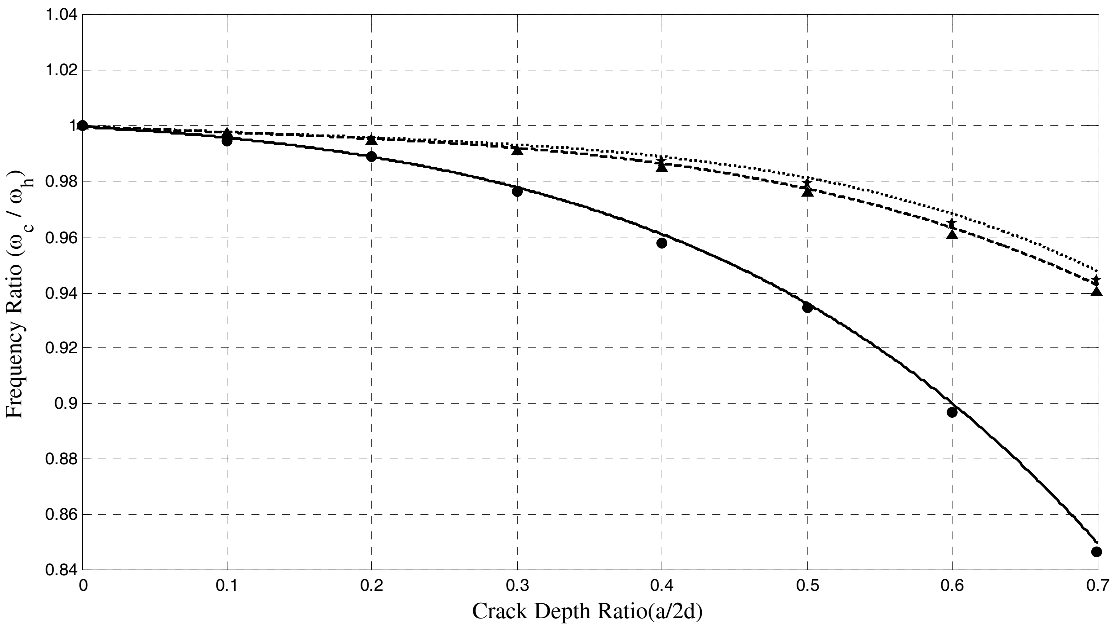

Figure 8. Fourth natural frequency ratio.

$x_C/L = 0.2$:analytical results(___) , finite element results(•); $x_C/L = 0.3$:analytical results(----) , finite element results(▲); $x_C/L = 0.4$:analytical results(….) , finite element results(★);

These results show that the natural frequencies will reduce due to the crack and by increasing the crack depth the natural frequencies will decrease more. Furthermore, the coincidence between finite element and analytical results can clearly be seen in these figures. Therefore the obtained model can use as a continuous model for flexural vibrations an Euler-Bernoulli beam with an open edge crack.

The interesting result of this research is the relation between the change in natural frequency of a crack beam and the crack position. The rate of change in first natural frequency of the beam increases while the crack moving toward the center of the beam (figure 5). But this is not true for other natural frequencies.

Another important result is the low sensitivity of the natural frequency of a beam to the crack. A crack which reduces the cross section of the beam up to 70% only causes a reduction less than 30% in first natural frequency and less than 10% in second natural frequency (figures 5 and 6).

7. CONCLUSION

A new continuous model for the bending vibration of a cracked beam has been developed in this paper. The governing equations of motion and corresponding boundary conditions have been obtained too. The crack effect is modeled as the crack disturbance function which leads to a modified displacement field. This function can be obtained by the fracture mechanics approach and the related parameters can be obtained by integration of the strain energy density over the cracked area.

The results show that the natural frequencies of a cracked beam reduce by increasing crack depth and the reduction rate depends on the crack depth.

The validity of this model has been investigated by comparing the analytical results with finite element results and a good coincidence between these two approaches has been observed.

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