TECHNICAL NOTE

CONTROLLABLE MULTI-SERVER QUEUE WITH BALKING

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Abstract This investigation proposes Markov chain analysis of controllable multi-server queue with balking. We incorporate an additional server which is added and removed at pre-specified threshold levels of queue size to control the balking behaviour of the customers. The steady state equations are constructed by assuming interdependent arrival and service processes which are characterized by bivariate Poisson distribution. Recursive approach is employed to establish the queue size distribution, which is further used to determine various performance indices.

Key words Controllable queue, Multi-server, Markov chain, Bivariate Poisson distribution, Balking, Additional server, Queue size.

چکیده   این مقاله یک صف چند سرویس دهنده ای قابل کنترل را بر مبنای زنجیره ای "مارکوف" تجزیه و تحلیل می نماید. به منظور کنترل رفتار مشتریان از یک سرویس دهنده اضافی، که با توجه به سطح استانه از بیش تعیین شده اضافه یا کم می شود، استفاده می گردد. معادلات حالت پایدار، با فرض ورودی های وابسته و پردازش های صف، توسط توزیع "پواسان" دو متغیره ایجاد می شود. برای توزیع اندازه صف -که تعیین کننده شاخص های کارآیی است- دیدگاه بازگشتی به کار گرفته شده است.

1. INTRODUCTION

In the past, many researchers have done significant works on multi-server queue with discouragement. In the queueing system with discouragement, the customers may either not like to join the queue or may leave the system without getting service due to impatience after joining it. In case of long queue, the balking behaviour of the customers can be controlled by improving the service facility. In the present study, we investigate a controllable multi-server queue with balking in which the service rate is controlled to decrease the queue length and to reduce the balking behaviour of the customers in the system by incorporating an additional server after a threshold queue size level.

With the increased interest in computer aided manufacturing, flexible manufacturing systems and manufacturing automation, industrial organizations posed challenging new problems regarding the control and design issues, which can be successfully managed to higher levels of efficiency and productivity via queue theoretic approaches. The present investigation provides the means to users to examine the effect of interdependent arrival and service processes for multi server queue, and combines and unifies the concerns for controllable rates and balking. The use of controllable and dependent rates make our model more versatile and closer to real time systems for automated manufacturing, computer controlled production systems, etc.

Queueing models with discouragement have been discussed by several researchers. The stationary solution of a multi server model with discouragement was studied by Reynolds (1968). Hiller and Libermann (1985) discussed various aspects of balking and reneging in their textbook "Operations Research". Analytical solution of the truncated inter-arrival Erlangian queue with balking and reneging was studied by Abou-El-Alta and Al Seedy (1985). Abou-El-Alta (1985) investigated truncated Poisson queue with balking, reneging and heterogeneity.

Abou-El-Alta (1991) developed state dependent M/M/1 queue with reneging and general balk functions. Wang (1993) investigated an M/M/c queueing model with (R, r) switch over policy. Interrelationship between queueing model with balking and reneging and machine repair problem with warm spares was studied by Gupta (1994). Singh and Kumar (1995) developed two queues in series with balking. Chauhan and Sharma (1996) considered profit analysis of M/M/R queueing model with balking and reneging. Jain (1998) investigated M/M/m queue with discouragement and additional servers. Jain (2000) suggested a general model of long queue with reneging and balking phenomenon. Jain and Singh (2002) analysed the M/M/m queueing with balking, reneging and additional servers. Recently N-policy for a machine repair system with spares and reneging was studied by Jain et al. (2004). Al-Seedy (2004) considered the queue with fixed and variable channels including balking and reneging concepts.

Most of the works on queueing models are based on the assumptions of independent input and service rates. However in many realistic congestion situations, this is not the case, so interdependent input and service rates must be taken into consideration. Except for a few Markovian models as cited above, the concept of interdependent rates is rarely employed; in particular for finite queue with balking, it is not mentioned in existing literature. In some special situations it is essential to treat interdependent arrival and service processes. Srinivasa Rao et al. (2000) investigated the M/M/1 interdependent queueing model with controllable arrival rates. The M/M/c inter dependent queueing model with controllable arrival rates was studied by Begum and Maheswari (2002). For double-ended queueing models having queues for both customers and servers, the controllable rates were used to analyze batch arrival queue by Sharma et al. (2004). Jain et al. (2004) studied machine repair problem with the provision of mixed standbys by incorporating controllable rates for failure and repair.

In this paper we develop a Markov model for controllable multi-server queue in which balking behaviour of the customers is taken into consideration. Due to balking, the customer may not join the queue, seeing it very long. The present study deals with the concept that the service rates can be controlled to decrease the queue length so as to discourage the balking behaviour of the customers. This is done by facilitating additional removable server when queue size reaches to a threshold level R; the additional server is withdrawn when queue size level ceases to r. The provision of an additional server is made which was not studied by previous workers for controllable queues. This may be beneficial to reduce the balking behaviour of the customers. The organization of the paper is as follows. Queueing model is described in section 2 by stating the arrival and service rates of both finite capacity and finite population models. The mathematical formulation and analysis of the both models are provided in sections 3 and 4, respectively. Performance measures namely mean queue length, expected waiting time, etc. for finite capacity and finite population models are given in section 5. Some special cases are discussed in section 6. Concluding remarks are given in final section 7.

2 MODEL DESCRIPTIONS

Consider Markovian queueing model with interdependent transition rates. The service facility consists of r permanent servers and single additional removable server. The additional server turns on when system queue size reaches to a pre-specified level (R) and removed when queue size ceases to a threshold level (r). The customers are assumed to balk with probability $1-\beta_1$, when all permanent servers are busy but additional server is not turned on. In case when all permanent as well as additional servers are busy, the customers are assumed to balk with probability $1-\beta_2$. The system is in state (n, i); n=0,1,2,…, L; i=0,1 when there are n customers in the system and service facility is working without additional server (i.e. i=0) or with additional server (i.e. i=1).

To formulate finite capacity and finite population models, the interdependent as well as state dependent rates are defined as follow:

The state dependent arrival rate for Finite Capacity Model (FCM) is given by

$$\lambda(n) = \begin{cases} \lambda - e; & n \le c, \ i = 0 \\ (\lambda - e)\beta_1; & c < n < R-1, \ i = 0 \\ (\lambda - e)\beta_2; & r+1 \le n < K, \ i = 1 \end{cases}$$

The arrival rate for Finite Population Model (FPM) is given by

$$\lambda(n) = \begin{cases} (M-n)\lambda - e; & n \le c, \ i = 0 \\ (M-n)(\lambda - e)\beta_1; & c < n < R-1, \ i = 0 \\ (M-n)(\lambda - e)\beta_2; & r+1 \le n < M, \ i = 1 \end{cases}$$

The service rate for both finite capacity and finite population models is given by

$$\mu(n) = \begin{cases} n(\mu - e); & n \le c, \ i = 0 \\ c(\mu - e); & c \le n < R-1, \ i = 0 \\ c(\mu - e) + (\mu_1 - e); & r+1 \le n < L, \ i = 1 \end{cases}$$

where L takes value K and M, respectively for finite capacity and finite population models. Let $P_n(i)$, i = 0,1 denote the steady state probability that there are n customers present in the system.

3. FINITE CAPACITY MODEL

In the finite capacity case, the steady state equations governing the model are given as follows:

$$-(\lambda - e)P_0(0) + (\mu - e)P_1(0) = 0 \qquad (1)$$

$$-\{(\lambda - e) + n(\mu - e)\}P_n(0) + (\lambda - e)P_{n-1}(0) + (n+1)(\mu - e)P_{n+1}(0) = 0, \quad 1 < n \le c-1 \qquad (2)$$

$$-\{(\lambda - e) + c(\mu - e)\}P_c(0) + (\lambda - e)P_{c-1}(0) + c(\mu - e)P_{c+1}(0) = 0 \qquad (3)$$

$$-\{(\lambda - e)\beta_1 + c(\mu - e)\}P_n(0) + (\lambda - e)\beta_1 P_{n-1}(0) + c(\mu - e)P_{n+1}(0) = 0, \qquad c < n \le r \qquad (4)$$

$$-\{(\lambda - e)\beta_1 + c(\mu - e)\}P_r(0) + (\lambda - e)\beta_1 P_{r-1}(0) + c(\mu - e)P_{r+1}(0) + \{c(\mu - e) + (\mu_1 - e)\}P_{r+1} = 0 \qquad (5)$$

$$-\{(\lambda - e)\beta_1 + c(\mu - e)\}P_n(0) + (\lambda - e)\beta_1 P_{n-1}(0) + c(\mu - e)P_{n+1}(0) = 0, \qquad r+1 < n \le R-2 \qquad (6)$$

$$-\{(\lambda - e)\beta_1 + c(\mu - e)\}P_{R-1}(0) + (\lambda - e)\beta_1 P_{R-2}(0) = 0 \qquad (7)$$

$$-\{(\lambda - e)\beta_2 + c(\mu - e) + (\mu_1 - e)\}P_{r+1}(1) + \{c(\mu - e) + (\mu_1 - e)\}P_{r+2}(1) = 0 \qquad (8)$$

$$-\{(\lambda - e)\beta_2 + c(\mu - e) + (\mu_1 - e)\}P_n(1) + (\lambda - e)\beta_2 P_{n-1}(1) + \{c(\mu - e) + (\mu_1 - e)\}P_{n+1}(1) = 0, \quad r+2 < n \le R-1 \qquad (9)$$

$$-\{(\lambda - e)\beta_2 + c(\mu - e) + (\mu_1 - e)\}P_R(1) + (\lambda - e)\beta_2 P_{R-1}(1) + \{c(\mu - e) + (\mu_1 - e)\}P_{R+1}(1) + (\lambda - e)\beta_1 P_{R-1}(0) = 0 \qquad (10)$$

$$-\{(\lambda - e)\beta_2 + c(\mu - e) + (\mu_1 - e)\}P_n(1) + (\lambda - e)\beta_2 P_{n-1}(1) + \{c(\mu - e) + (\mu_1 - e)\}P_{n+1}(1) = 0, \qquad R+1 \le n < K-1 \qquad (11)$$

$$-(\lambda - e)\beta_2 P_{K-1}(1) + \{c(\mu - e) + (\mu_1 - e)\}P_K(1) = 0, \qquad (12)$$

From equations (1) and (2), the expressions for $P_n(0)$, $(0 \le n \le c)$ are obtained as

$$P_n(0) = \left(\frac{\lambda - e}{\mu - e}\right)^n \frac{1}{n!} P_0(0) \qquad (13)$$

The expressions for $P_n(0)$ $(c \le n \le r)$ are recursively derived from the equation (4) as

$$P_n(0) = \left(\frac{\lambda - e}{\mu - e}\right)^n \left(\frac{\beta_1}{c}\right)^{n-c} \frac{1}{c!} P_0(0) \qquad (14)$$

From equations (5) and (6), the expression for $P_n(0)$ $(r+1 \le n \le R-1)$ is given by

$$P_n(0) = \left(\frac{\lambda - e}{\mu - e}\right)^n \left(\frac{\beta_1}{c}\right)^{r-c+1} \frac{1}{c!} P_0(0) - D\left[\frac{1-B^{n-r}}{1-B}\right]P_{r+1}(1), \qquad (15)$$

where $D = \left\{1 + \left(\frac{\mu_1 - e}{c(\mu - e)}\right)\right\}$, $B = \left(\frac{(\lambda - e)\beta_1}{c(\mu - e)}\right)$

and $A = \left(B^r - B^{-R}\right)^{-1}$

Solving equation (6), we get

$$P_{r+1}(1) = B^R \left(\frac{\beta_1}{c}\right)^{-c} \frac{1}{c!} A D^{-1} B^r (1-B) P_0(0) \qquad (16)$$

Substituting result from equation (15) in equation (14), we get

$$P_n(0) = B^r \left(B^n - B^R\right) A \left(\frac{\beta_1}{c}\right)^{-c} \frac{1}{c!} P_0(0), \quad r+1 < n \le R-1 \qquad (17)$$

Thus the expression for $P_n(0)$ $(0 \le n \le R-1)$ is given in term of $P_0(0)$. The remaining probabilities $P_n(1)$ for $n \ge r+1$ are calculated from the equations (8) to (12) as obtained below:
From equations (8) and (9), the expressions for $P_n(1)$ is obtained as

$$P_n(1) = \left(\frac{1 - E^{n-r}}{1-E}\right) P_{r+1}(1) \qquad (18)$$

where $E = \dfrac{(\lambda - e)\beta_2}{c(\mu - e) + (\mu_1 - e)}$

From equations (10) and (11), the expression for $P_n(1)$, $(R \le n < K)$ is derived as

$$P_n(1) = \frac{E^n \left(E^{-R} - E^{-r}\right)}{1 - E} P_{r+1}(1) \qquad (19)$$

From equation (12), we get

$$P_K(1) = \frac{E\left(E^{-R} - E^{-r}\right)}{1 - E} P_{r+1}(1) \qquad (20)$$

Substituting the values of $P_{r+1}(1)$ from equation (16) in equations (18)-(20), we determine all the steady state probabilities which are expressed in term of $P_0(0)$.

4. FINITE POPULATION MODEL

In the finite capacity model, the steady state equations governing the model are given as follows:

$$-\{(M-n)(\lambda - e) + n(\mu - e)\}P_n(0) + (M - \overline{n-1})(\lambda - e)P_{n-1}(0) + (n+1)(\mu - e)P_{n+1}(0) = 0, \; 1 < n \le c-1 \qquad (21)$$

$$-\{(M-n)(\lambda - e) + n(\mu - e)\}P_n(0) + (M - \overline{n-1})(\lambda - e)P_{n-1}(0) + (n+1)(\mu - e)P_{n+1}(0) = 0, \; 1 < n \le c-1 \qquad (22)$$

$$-\{(M-c)(\lambda - e) + c(\mu - e)\}P_c(0) + (M - \overline{c-1})(\lambda - e)P_{c-1}(0) + c(\mu - e)P_{c+1}(0) = 0 \qquad (23)$$

$$-\{(M-n)(\lambda - e)\beta_1 + c(\mu - e)\}P_n(0) + (M - \overline{n-1})(\lambda - e)\beta_1 P_{n-1}(0) + c(\mu - e)P_{n+1}(0) = 0, \quad c < n \le r \qquad (24)$$

$$-\{(M-r)(\lambda - e)\beta_1 + c(\mu - e)\}P_r(0) + (M - \overline{r-1})(\lambda - e)\beta_1 P_{r-1}(0) + c(\mu - e)P_{r+1}(0) + \{c(\mu - e) + (\mu_1 - e)\}P_{r+1} = 0 \qquad (25)$$

$$-\{(M-n)(\lambda - e)\beta_1 + c(\mu - e)\}P_n(0) + (M - \overline{n-1})(\lambda - e)\beta_1 P_{n-1}(0)] + c(\mu - e)P_{n+1}(0) = 0, \quad r+1 < n \le R-2 \qquad (26)$$

$$- \left\{ \left(M - \overline{R-1} \right)(\lambda - e)\beta_1 + c(\mu - e) \right\} P_{R-1}(0) +$$
$$\left(M - \overline{R-2} \right)(\lambda - e)\beta_1 P_{R-2}(0) = 0 \quad (27)$$

$$- \left\{ \left(M - \overline{r+1} \right)(\lambda - e)\beta_2 + c(\mu - e) + (\mu_1 - e) \right\}$$
$$\times P_{r+1}(1) + \left\{ c(\mu - e) + (\mu_1 - e) \right\} P_{r+2}(1) = 0 \quad (28)$$

$$- \left\{ (M - n)(\lambda - e)\beta_2 + c(\mu - e) + (\mu_1 - e) \right\} P_n(1) + \left(M - \overline{n-1} \right)$$
$$\times (\lambda - e)\beta_2 P_{n-1}(0) + \left\{ c(\mu - e) + (\mu_1 - e) \right\} P_{n+1}(1) = 0, \quad r+2 < n \le R-1 \quad (29)$$

$$- \left\{ (M - R)(\lambda - e)\beta_2 + c(\mu - e) + (\mu_1 - e) \right\}$$
$$\times P_R(1) + \left(M - \overline{R-1} \right)(\lambda - e)\beta_2 P_{R-1}(1) \quad (30)$$
$$+ \left\{ c(\mu - e) + (\mu_1 - e) \right\} P_{R+1}(1) + (\lambda - e)\beta_1 P_{R-1}(0) = 0$$

$$- \left\{ (M - n)(\lambda - e)\beta_2 + c(\mu - e) + (\mu_1 - e) \right\}$$
$$\times P_n(1) + (M - n)(\lambda - e)\beta_2 P_{n-1}(1) \quad (31)$$
$$+ \left\{ c(\mu - e) + (\mu_1 - e) \right\} P_{n+1}(1) = 0, \quad R+1 \le n < M-1$$

$$- \left(M - \overline{K-1} \right)(\lambda - e)\beta_2 P_{M-1}(1) + \left\{ c(\mu - e) + (\mu_1 - e) \right\} P_M(1) = 0 \quad (32)$$

From equations (21) and (22), the expression for $P_n(0)$, $\left(0 \le n \le c \right)$ are obtained as

$$P_n(0) = \frac{M!}{(M-n)!} \left(\frac{\lambda - e}{\mu - e} \right)^n \frac{1}{n!} P_0 \quad (33)$$

The expression for $P_n(0)$ $\left(c \le n \le r \right)$ is recursively derived from the equation (24) as

$$P_n(0) = \frac{M!}{(M-n)!} \left(\frac{\lambda - e}{\mu - e} \right)^n \left(\frac{\beta_1}{c} \right)^{n-c} \frac{1}{c!} P_0(0) \quad (34)$$

From equations (25) and (26), we obtain $P_n(0)$ as

$$P_n(0) = \frac{M!}{(M-n)!} \left(\frac{\lambda - e}{\mu - e} \right)^n \left(\frac{\beta_1}{c} \right)^{n-c} \frac{1}{c!} P_0(0)$$
$$- \left[1 + \sum_{i=1}^{n-r-1} B^i (M-n+1)_i \right] D P_{r+1}(1)$$
$$(r+1 \le n \le R-1) \quad (35)$$

Here we use the Pochhammer's symbol, i.e.
$(a)_0 = 1$, $(a)_n = a(a+1)\dots\{a+(n-1)\}$.

and $D = \left\{ 1 + \left(\frac{\mu_1 - e}{c(\mu - e)} \right) \right\}$.

Equation (26) gives,

$$P_{r+1}(1) = \frac{M!}{(M-R)!} \left(\frac{\lambda - e}{\mu - e} \right)^R \left(\frac{\beta_1}{c} \right)^{R-c}$$
$$\times \frac{1}{c!} D^{-1} H P_0(0), \quad r+1 < n \le R-1 \quad (36)$$

where $H = 1 / \left[1 + \sum_{i=1}^{R-r-1} B^i (M-R-1)_i \right]$

Substituting this value of $P_{r+1}(1)$ in equation (35), we get

$$P_n(0) = \frac{1}{c!} \left[\frac{M!}{(M-n)!} B^n - \frac{M!}{(M-R)!} B^r \right] \left(\frac{\beta_1}{c} \right)^{-c} P_0(0) \quad (37)$$

Thus the expression for $P_n(0)$, $\left(0 \le n \le R-1 \right)$ is obtained in terms of $P_0(0)$. The remaining probabilities $P_n(1)$ for $n \ge r+1$ are calculated from the equations (28) to (32) as follows.

From equations (28) and (29), we get

$$P_n(0) = \left[1 + \sum_{i=1}^{n-r-1} B^i (M-n+1)_i \right] P_{r+1}(1) \quad (38)$$

where $E = \frac{(\lambda - e)\beta_2}{c(\mu - e) + (\mu_1 - e)}$.

Equations (30) and (31) give the expression for $P_n(1)$, $(R \le n < M)$ as

$$P_n(1) = \begin{bmatrix} 1 - U_{n-R} + \{ 1 + (M-R)U_{n-R} \} S_{R-1} \\ -(M-R+1)E^2 U_{n-R} \end{bmatrix} P_{r+1}(1) - E U_{n-R} Q P_0(0) \quad (39)$$

where $U_i = \sum_{j=0}^{i-1} \delta_j E^{j+1}$, $\delta_j = \left(M - \overline{R+1} \right)_j$,

$$Q = \frac{1}{c!} \left[\frac{M!}{(M-R-1)!} B^{R-1} - \frac{M!}{(M-R)!} B^r \right] \left(\frac{\beta_1}{c} \right)^{-c}$$

$$T = \frac{(\lambda - e)\beta_1}{c(\mu - e) + (\mu_1 - e)},$$

$$S_q = \sum_{i=1}^{q-r} B^i (M-q)_i$$

Equation (32) gives the expression for $P_M(1)$ as:

$$P_M(1) = E \begin{bmatrix} 1 - U_{M-1-R} + \{1 + (M-R)U_{M-1-R}\}S_{R-1} \\ -(M-R+1)E^2 U_{M-1-R} \end{bmatrix} P_{r+1}(1) - EU_{M-1-R}Q$$

(40)

Using equations (36), (38)-(40) all the steady state probabilities are determined in term of $P_0(0)$.

5. PERFORMANCE MEASURES

After obtaining queue size distribution in previous section, we derive results for queue length and expected waiting time as follows:

5.1 Finite Capacity Model
(i) Mean queue length
The average number of customers present in the system is given by

$$L = \begin{bmatrix} \sum_{n=0}^{c}\left(\frac{\lambda-e}{\mu-e}\right)^n \frac{1}{(n-1)!} + \sum_{n=c+1}^{r} n\left(\frac{\lambda-e}{\mu-e}\right)^n\left(\frac{\beta_1}{c}\right)^{n-c}\frac{1}{c!} + \\ \sum_{n=r+1}^{R-1} nB^n(B^n - B^R)A\left(\frac{\beta_1}{c}\right)^{-c}\frac{1}{c!} \\ + \sum_{n=r+1}^{R} n\left(\frac{1-E^{n-r}}{1-E}\right)B^R\left(\frac{\beta_1}{c}\right)^{-c}\frac{1}{c!}AD^jB(1-B) \\ + \sum_{n=R+1}^{K} n\frac{E(E^{-R}-E^{-r})}{1-E}B^R\left(\frac{\beta_1}{c}\right)^{-c}\frac{1}{c!}AD^jB(1-B) \end{bmatrix} P_0(0)$$

(41)

(ii)Expected waiting time

To calculate the expected waiting time of the customers in the system we use the Little formula

i.e. $W = \dfrac{L}{\lambda_{eff}}$

(42)

where λ_{eff} is the effective arrival rate of the system and is given by.

$$\lambda_{eff} = \lambda\left(\sum_{n=0}^{c} P_n + \beta_1\sum_{n=c+1}^{R-1} P_n + \beta_2\sum_{n=r+1}^{K} P_n\right)$$

(43)

5.2 Finite Population Model
(i) Mean queue length
For this model, the average number of customers present in the system is given by

$$L = [\sum_{n=0}^{c} \frac{M!}{(M-n)!}\left(\frac{\lambda-e}{\mu-e}\right)^n \frac{1}{(n-1)!}$$

$$+ \sum_{n=c+1}^{r} \frac{M!}{(M-n)!}\left(\frac{\lambda-e}{\mu-e}\right)^n\left(\frac{\beta_1}{c}\right)^{n-c}\frac{1}{c!}$$

$$+ \sum_{n=r+1}^{R-1} \frac{n}{c!}\left[\frac{M!}{(M-n)!}B^n - \frac{M!}{(M-R)!}B^r\right]\left(\frac{\beta_1}{c}\right)^{-c}$$

$$+ \sum_{n=r+1}^{R} \frac{n}{c!}\left[1 + \sum_{i=1}^{n-r-1} B^i(M-n+1)_i\right]$$

$$\times \frac{M!}{(M-R)!}\left(\frac{\lambda-e}{\mu-e}\right)^R\left(\frac{\beta_1}{c}\right)^{R-c} D^{-1}H]P_0(0)$$

$$+ \sum_{n=R+1}^{K} n\begin{bmatrix} (M-\overline{k-1})E\begin{bmatrix} 1 - U_{M-1-R} \\ +\{1+(M-R)U_{M-1-R}\}S_{R-1} \\ -(M-R+1)E^2 U_{M-1-R}\end{bmatrix}P_{r+1}(1) \\ -EU_{M-1-R}\end{bmatrix}$$

(44)

(ii) Expected waiting time
The expected waiting time of the customers in the system is obtained by using the Little's formula and is given by

$$W = \frac{L}{\lambda_{eff}}$$

(45)

where

$$\lambda_{eff} = \lambda \left(\sum_{n=0}^{c} P_n + \beta_1 \sum_{n=c+1}^{R-1} P_n + \beta_2 \sum_{n=r+1}^{M} P_n \right)$$

(46)

6. SPECIAL CASES

Case I: *M/M/c controllable queueing model with balking.*
In this case additional server is not incorporated so that $\mu_i=0$.

Finite capacity model:
The expression for $P_n(0)$ $(0 \le n \le R-1)$ is given as follows:

$$P_n(0) = \begin{cases} \left(\dfrac{\lambda-e}{\mu-e}\right)^n \dfrac{1}{n!} P_0(0); & 0 \le n \le c \\[2ex] \left(\dfrac{\lambda-e}{\mu-e}\right)^n \left(\dfrac{\beta_1}{c}\right)^{n-c} \dfrac{1}{c!} P_0(0); & c \le n \le r \\[2ex] \left(\dfrac{\lambda-e}{\mu-e}\right)^n \left(\dfrac{\beta_1}{c}\right)^{r-c+1} \\[1ex] \times \dfrac{1}{c!} P_0(0) - \left[\dfrac{1-B^{n-r}}{1-B}\right] P_{r+1}(1); & r+1 \le n \le R-1 \\[2ex] B^r\left(B^n - B^R\right) A\left(\dfrac{\beta_1}{c}\right)^{-c} \dfrac{1}{c!} P_0(0); & r+1 < n \le R-1 \end{cases}$$

(47)

where $P_{r+1}(1) = B^R\left(\dfrac{\beta_1}{c}\right)^{-c} \dfrac{1}{c!} AB^r (1-B) P_0(0);$

$D = \left\{ 1 + \left(\dfrac{\mu_1-e}{c(\mu-e)} \right) \right\};$

$A = (B^r - B^{-R})^{-1}$

and

$B = \left(\dfrac{(\lambda-e)\beta_1}{c(\mu-e)} \right).$

The expressions for $P_n(1)$, $n \ge r+1$ is obtained as

$$P_n(1) = \begin{cases} \left[\dfrac{1-E^{n-r}}{1-E}\right] P_{r+1}(1); & r+1 \le n < R-1 \\[2ex] \dfrac{E^n\left(E^{-R} - E^{-r}\right)}{1-E} P_{r+1}(1); & R \le n < K \end{cases}$$

(48)

where

$$E = \frac{(\lambda-e)\beta_2}{c(\mu-e) + (\mu_1-e)}$$

$$P_K(1) = \frac{E\left(E^{-R} - E^{-r}\right)}{1-E} P_{r+1}(1) \quad \text{And}$$

(49)

Case II: *M/M/c controllable queueing model with additional server.*
When $\beta_1 = \beta_2 = 1$ then our model provides results for this case.

Finite capacity model:
The expression for $P_n(0)$ $(0 \le n \le R-1)$ is given as follows:

$$P_n(0) = \begin{cases} \left(\dfrac{\lambda-e}{\mu-e}\right)^n \dfrac{1}{n!} P_0(0); & 0 \le n \le c \\[2ex] \left(\dfrac{\lambda-e}{\mu-e}\right)^n \left(\dfrac{1}{c}\right)^{n-c} \dfrac{1}{c!} P_0(0); & c \le n \le r \\[2ex] \left(\dfrac{\lambda-e}{\mu-e}\right)^n \left(\dfrac{1}{c}\right)^{r-c+1} \dfrac{1}{c!} P_0(0) \\[1ex] - D\left[\dfrac{1-B^{n-r}}{1-B}\right] P_{r+1}(1); & r+1 \le n \le R-1 \\[2ex] B^r\left(B^n - B^R\right) A\left(\dfrac{1}{c}\right)^{-c} \dfrac{1}{c!} P_0(0); & r+1 < n \le R-1 \end{cases}$$

(50)

$P_{r+1}(1) = B^R\left(\dfrac{\beta_1}{c}\right)^{-c} \dfrac{1}{c!} AD^{-1}B^r (1-B) P_0(0);$

$A = (B^r - B^{-R})^{-1}$

and $B = \left(\dfrac{(\lambda-e)\beta_1}{c(\mu-e)} \right)$

The expression for $P_n(1)$, $n \geq r+1$ is obtained as:

$$P_n(1) = \begin{cases} \left(\dfrac{1-E^{n-r}}{1-E}\right)P_{r+1}(1); & r+1 \leq n < R-1 \\[4mm] \dfrac{E^n\left(E^{-R}-E^{-r}\right)}{1-E}P_{r+1}(1); & R \leq n < K \end{cases}$$

(51)

where $\quad E = \dfrac{(\lambda - e)}{c(\mu - e) + (\mu_1 - e)}$

and

$$P_K(1) = \frac{E\left(E^{-R}-E^{-r}\right)}{1-E}P_{r+1}(1)$$

(52)

Case III: *M/M/c controllable queueing model.*
In this case we put $\beta_1 = \beta_2 = 1$ and $\mu_1 = 0$ so that our model coincides with the model studied by Begum and Maheswari (2002).

Case IV: Substituting c=1 in case III, we obtain result for M/M/1 controllable queue studied by Srinivasa Rao et al. (2000).

Case V: When arrival and service rates are not interdependent, we put e=0, in above all four cases and obtain corresponding results. By this particular substitution case III and IV provide results for classical M/M/c and M/M/1 queueing models (cf. Hiller & Liberman, 1985).

7. DISCUSSIONS

In this paper, we have studied controllable multi-server queue with balking in which the service rates are controlled to decrease the queue length by incorporating the additional removable server. The analytical results derived may be employed to avoid the balking behaviour in many congestion situations in particular when arrival and service rates are interdependent. The provision of additional server after a pre-specified queue level is recommended to avoid excessive cost in many real life congestion situations e.g. at railway reservation booking windows, cafeteria, bank cash counters, etc.

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