

---

## RESEARCH NOTE

---

# A COMPUTATIONAL STUDY OF OXYGEN TRANSPORT IN THE BODY OF LIVING ORGANISM

*Sanjeev Kumar and Narendra Kumar*

*Department of Mathematics, Institute of Basic Science  
Dr. B.R. Ambedaker (Agra) University, Khandari, Agra, India, 282002  
sanjeevibs@yahoo.co.in*

(Received: May 2, 2003 – Accepted in Revised Form: December 6, 2004)

**Abstract** Oxygen is an essential part of the living organism. It is transported from blood to the body tissue by the systematic circulation and large part of it is stored in the blood flowing in capillaries. In this work we discuss a mathematical model for oxygen transport in tissues. The governing equations are established assuming that the blood is flowing along a co-axial cylindrical capillary inside the tissue and has a constant partial pressure of oxygen. We solve the governing partial differential equations using finite element techniques. The main object of the present work is to investigate the effects of various assumptions such as neglecting axial diffusion and neglecting the effect of facilitated myoglobin diffusion.

**Keywords** Axial Diffusion, Hemoglobin, Myoglobin, Co-axial Cylindrical Capillary

**چکیده** اکسیژن بخشی ضروری در ارگانیسم موجود زنده است. این ماده از طریق چرخش سیستماتیک خون در بدن انتقال می یابد. مقدار بزرگی از اکسیژن در خون جاری در مویرگ ها ذخیره شده و به بافت های موجود زنده انتقال می یابد. در این مقاله یک مدل ریاضی برای انتقال اکسیژن به بافت ها بحث می شود. معادلات حاکم با فرض اینکه خون در استوانه های موازی هم محور در داخل بافت با فشار جزئی ثابت اکسیژن در حرکت است، نوشته می شوند. معادلات دیفرانسیل پاره ای حاصل به روش المان محدود حل می شود. هدف اصلی این تحقیق تعیین نقش فرض های گوناگون مانند ندیده گرفتن نفوذ محوری و تاثیر تسهیل نفوذ توسط میوگلوبین بر انتقال اکسیژن است.

## 1. INTRODUCTION

Blood consists of many components such as (i) the red blood cells, which contain the protein molecule that acts as the carrier of oxygen and carbon dioxide in the human system, (ii) the white blood cells which ingest and destroy harmful bacteria's and dead cells and behave as the defense forces of the body against infection and injury, (iii) platelets that help in the clotting of blood and (iv) plasma which is the liquid part of the blood and contains several salts, glucose, amino acids, proteins, hormones etc. Here we are considering our study only on transport of oxygen in the blood. Oxygen is transported to the tissues by the systemic circulation. During its transport in capillary region

a small fraction of oxygen is dissolved in plasma. While in tissue region it is driven by pressure gradient. Oxygen stored in tissue may be dissolved or chemically bound to myoglobin. A large amount of oxygen stored in the blood flowing into the capillaries is chemically bound to hemoglobin. As the oxygen is transported from blood to tissue, therefore it involves convection, diffusion and reaction processes.

Schubert and Zhang [7] reported about the importance of axial diffusion in the experimental data through a model consisting of cylindrical capillary with blood flowing through it, and surrounded by a co-axial cylindrical tissue compartment, despite the common assumption of neglecting diffusion in the direction of blood flow,

Sharan et al [4] considered a single capillary surrounded by a co-axial cylindrical tissue compartment. In this capillary, diffusion is in both axial and radial directions. Many researchers have investigated the solution of the governing equations using various simplifications. Murray [2] considered the one-dimensional transport of oxygen through a solution containing either hemoglobin or myoglobin. Reaction terms, between oxygen and myoglobin in the tissue and between oxygen and hemoglobin in the blood, are taken of non-linear nature. This model considers diffusion in one dimension and reaction between oxygen and either hemoglobin or myoglobin molecule.

## 2. MODELING

The geometry of the present mathematical model is explained by Figure 1. Here blood flowing into the capillary from the artery is named as arterial blood. The equations governing oxygen transport in the capillary are:

$$a_p D_p \nabla^2 p = v a_p \frac{\partial p}{\partial z} - A_c \quad (1)$$

$$C_h D_h \nabla^2 H = v C_h \frac{\partial H}{\partial z} + A_c \quad (2)$$

Here  $A_c$  is the reaction term of hemoglobin with oxygen. As in Kapur [3]

$$A_c = \frac{f(p)}{1 + f(p)} \quad (3)$$

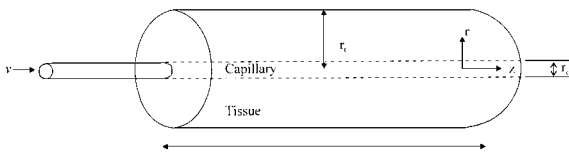


Figure 1. Geometry of mathematical model used.

Using the dissociation curve given by Kelman [1],

$$f(p) = \begin{cases} \frac{a_1 p + a_2 p^2 + a_3 p^3 + p^4}{a_4 + a_5 p + a_6 p^2 + a_7 p^3 + p^4} & p \leq 12 \\ .003683p + .000584p^2 & p > 12 \end{cases} \quad (4)$$

The constants used in equation (4) are given by

$$a_1 = 8.5322289 \times 10^3$$

$$a_2 = 2.1214010 \times 10^3$$

$$a_3 = -6.7073989 \times 10^2$$

$$a_4 = 9.3596087 \times 10^5$$

$$a_5 = -3.1346258 \times 10^4$$

$$a_6 = 2.3961674 \times 10^3$$

$$a_7 = -6.7104406 \times 10^3$$

When oxygen pressure and hemoglobin saturation are in equilibrium then reaction term (3) in the governing equations will vanishes. Governing equations in tissue region are:

$$a_t D_t \nabla^2 p = -A_t + Q \quad (5)$$

$$C_m D_m \nabla^2 M = -A_t \quad (6)$$

Here ' $A_t$ ' is reaction term between oxygen and myoglobin and is given by

$$A_t = u_m C_m M - a_t u_m^1 C_m (1 - M)p \quad (7)$$

and in equilibrium, the dissociation curve for myoglobin saturation is

$$g(p) = \frac{u'_m a_t p}{u'_m a_t p + u_m} \quad (8)$$

$A_t$  the reaction term, (after combining both the equations (7) and (8)), can be written as

$$A_t = C_m u_m \frac{M - g(p)}{1 - g(p)} \quad (9)$$

Boundary conditions for both the set of equations are given below:

(i) No diffusion flux of Myoglobin outside the tissue

$$\frac{\partial M}{\partial r} = 0, \quad r = r_c, r_t \quad 0 \leq z \leq b$$

$$\frac{\partial M}{\partial z} = 0, \quad z = 0, b \quad r_c \leq r \leq r_t$$

(ii) No diffusion flux of hemoglobin outside capillary across any other boundary

$$\frac{\partial H}{\partial r} = 0, \quad r = r_c, \quad 0 \leq z \leq b$$

$$\frac{\partial H}{\partial z} = 0, \quad z = a, \quad 0 \leq r \leq r_c$$

(iii)  $p$  and the flux of oxygen are continuous across the capillary tissue interface, therefore the condition are:

$$p \Big|_{z=r_c^-} = p \Big|_{z=r_c^+} \quad 0 \leq z \leq b$$

$$a_p D_p \frac{\partial p}{\partial r} \Big|_{z=r_c^-} = a_t D_t \frac{\partial p}{\partial r} \Big|_{z=r_c^+} \quad 0 \leq z \leq b$$

(iv) Hemoglobin saturation is in equilibrium with partial pressure of oxygen in blood entering the capillary

$$H = f(p_a) \quad ; z = 0 \quad 0 \leq r \leq r_c$$

(v) The flux-dissolved oxygen per unit area is given by  $D\nabla$ , and is zero across closed boundaries. In addition to that we assume a non-diffusion flux condition where blood flows out of the capillary and so

$$\frac{\partial p}{\partial z} = 0 \quad z = 0, \quad r_c \leq r \leq r_t$$

$$\frac{\partial p}{\partial r} = 0 \quad r = r_t, \quad 0 \leq z \leq b$$

$$\frac{\partial p}{\partial z} = 0 \quad z = b, \quad 0 \leq r \leq r_t$$

(vi) Radial symmetry:

$$\frac{\partial p}{\partial r} = \frac{\partial H}{\partial r} = 0 \quad \text{at } r = 0, \quad 0 \leq z \leq b$$

(vii)  $P$  is initially in equilibrium with arterial blood:

$$p = p_a \quad z = 0, \quad 0 \leq r \leq r_c$$

(viii) For a unique solution of fractional myoglobin saturation  $M$ , we have:

$$M(b, r_t) = g(p(b, r_t))$$

In the next step we non-dimensionalize the model equations, where we choose scaling such that the non-dimensional variables range from 0 to 1. Here  $L$  is the length scale to be chosen appropriately.

$$\left. \begin{aligned} p &\rightarrow p_0 + (p_1 - p_0)p \\ H &\rightarrow H_0 + (H_1 - H_0)H \\ r &\rightarrow r_c r \\ z &\rightarrow L_z z \end{aligned} \right\} \quad (10)$$

then equation (1) and (2) turn to the equations

$$A_1 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + A_2 \frac{\partial^2 p}{\partial z^2} = \frac{\partial p}{\partial z} + A_3 \frac{f(p)}{(1-H_0)/(H_1 H_0) + f(p)} \quad (11)$$

$$B_1 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial H}{\partial r} \right) + B_2 \frac{\partial^2 H}{\partial z^2} = \frac{\partial H}{\partial z} - B_3 \frac{f(p)}{(1-H_0)/(H_1 H_0) + f(p)} \quad (12)$$

Neglecting the terms of  $O(10^{-2})$  the size of the largest terms,

$$A_1 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + A_2 \frac{\partial^2 p}{\partial z^2} = \frac{\partial p}{\partial z} + A_3 \frac{f(p)}{1 + f(p)} \quad (13)$$

$$0 = \frac{\partial H}{\partial z} - B_3 \frac{f(p)}{1 + f(p)} \quad (14)$$

With the help of (13) and (14) we have

$$A_1 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + A_2 \frac{\partial^2 p}{\partial z^2} = \left( 1 + \frac{A_3}{B_3} f'(p) \right) \frac{\partial p}{\partial z} \quad (15)$$

### 3. TISSUE REGION

Using the parameters

$$p \rightarrow p_0 + (p_1 - p_0)p,$$

$$M \rightarrow M_0 + (m_1 - m_0)M$$

$$r \rightarrow L_r r \quad z \rightarrow L_z z \quad (16)$$

equation (5) and (6) turn to

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{L_r^2}{L_z^2} \frac{\partial^2 p}{\partial z^2} = -C_1 \frac{g(p)}{\frac{1-M_0}{M_1-M_0} + g(p)} + C_2 \quad (17)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial M}{\partial r} \right) + \frac{L_r^2}{L_z^2} \frac{\partial^2 M}{\partial z^2} = -D_1 \frac{g(p)}{\frac{1-M_0}{M_1-M_0} + g(p)} + D_2 \quad (18)$$

On neglecting terms of size  $O(10^{-2})$ , the magnitude from equation (18) is given by

$$D_1 \frac{g(p)}{1 + g(p)} = 0 \quad (19)$$

from which we deduce that we can assume an instantaneous reaction and  $g(p) = 0$ .

### 4. RESULTS AND DISCUSSION

For the purpose of our numerical calculation we used the following values different terms used in above model

$$r_c = 3.3 \times 10^{-4} \text{ cm} \quad r_t = 3.3 \times 10^{-3} \text{ cm}$$

$$\begin{aligned} b &= 1.6 \times 10^2 \text{ cm} & v &= 0.02 \text{ unit} \\ D_p &= 1.0 \times 10^{-5} \text{ unit} & D_t &= 1.6 \times 10^{-5} \text{ unit} \\ D_h &= 1.4 \times 10^{-7} \text{ unit} & a_p &= 1.53 \times 10^{-9} \text{ unit} \\ D_m &= 5.0 \times 10^{-7} \text{ unit} & C_h &= 8.9 \times 10^{-6} \text{ unit} \\ C_m &= 2.7 \times 10^{-7} \text{ unit} & U_h &= 60 \text{ unit} \\ U_h' &= 2 \times 10^{10} \text{ unit} & Q &= 4.5 \times 10^{-8} \text{ unit} \\ a_t &= 1.3 \times 10^{-9} \text{ unit} \end{aligned}$$

Based on these results, we came to know that the axial diffusion term should not be neglected as many workers have neglected it in their solution of the governing equations. In tissue region, axial diffusion is nearly 15 times smaller than the radial diffusion and so should be avoided by equation (17) and (18). This is an indication for the importance of axial diffusion. As the ratio  $(L_r/L_z) \in [0,1]$  increases axial diffusion term becomes large. Salathe et al [5] have also investigated the boundary layers in the region of the inflow boundary using matched asymptotic expansion.

On the basis of our results, we also observed that on increasing consumption of oxygen,  $p$  is lowered; therefore we cannot neglect myoglobin because myoglobin plays an emergency storehouse of oxygen in case of deficiency of oxygen (Figures 2 to 5).

Finally, we came to know that the transport is a sort of convection dominated in the axial direction inside the capillary. This will be a small order diffusive flux of oxygen along the inflow tube and so the assumption  $p = p_a$  for blood inflow with geometric basis is valid. Also in case of any flow geometry we can assume that the diffusion is of neglecting order.

### 5. SYMBOLS USED IN MODEL

- R** – Radial co-ordinate
- $\Delta_p$  – Diffusion coefficient of oxygen in plasma
- p** – Oxygen partial pressure
- H** – Fractional hemoglobin saturation
- Q<sub>p</sub>** – Oxygen solubility in plasma
- $\Delta_t$  – Diffusion coefficient of oxygen in tissue
- D<sub>p</sub>** – Diffusion coefficient of hemoglobin

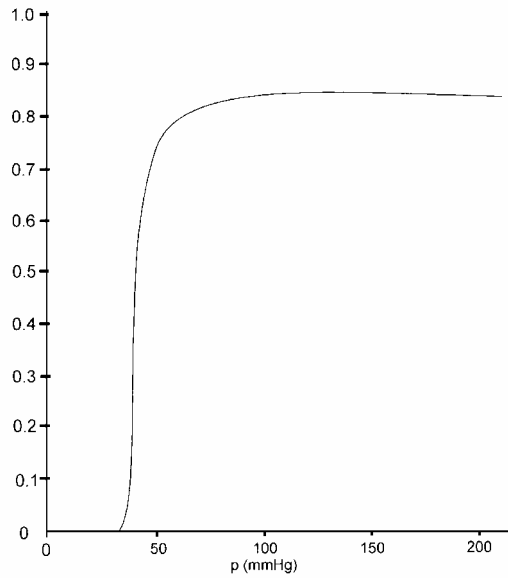


Figure 2. The oxyhaemoglobin dissociation curve.

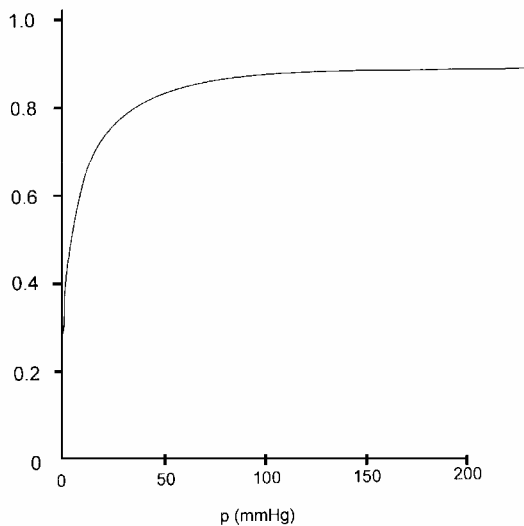


Figure 3. The oxyhaemoglobin dissociation curve.

$D_m$  – Diffusion coefficient of myoglobin  
 $C_h$  – Oxygen carrying capacity of hemoglobin  
 $r_c$  – Capillary radius  
 $C_m$  – Oxygen carrying capacity of myoglobin  
 $b$  – Length of capillary  
 $U_h$  – Backward reaction rate for oxy-hemoglobin reaction.  
 $r_t$  – Tissue radius  
 $U_m$  – Backward reaction rate for oxy-myoglobin reaction

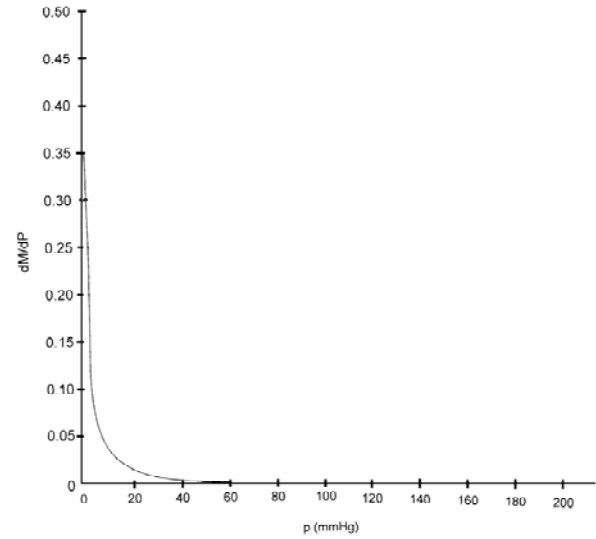
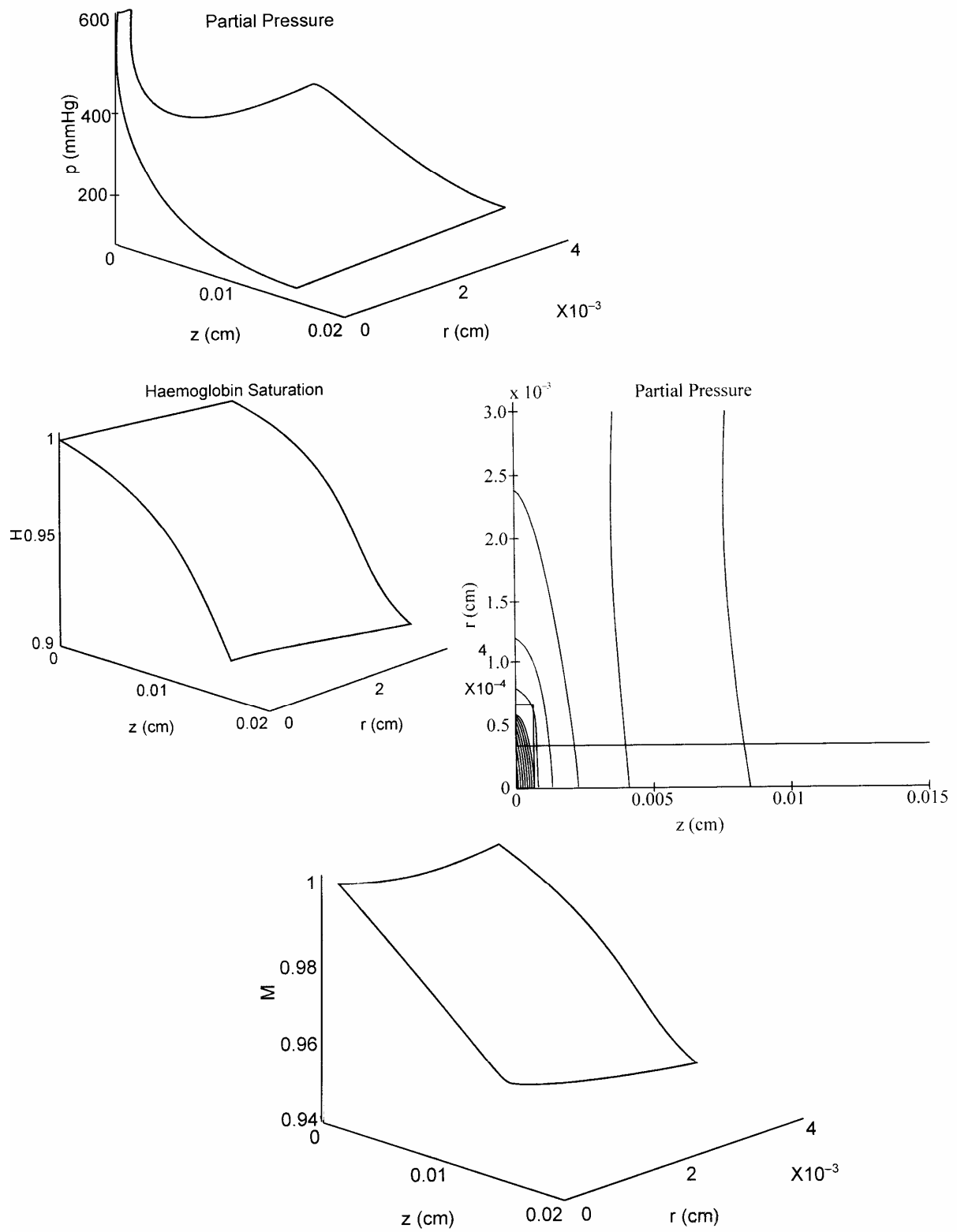


Figure 4. The derivative of the myoglobin dissociation curve.

$Q_t$  – oxygen solubility in tissue

## 6. REFERENCES

1. Kelman, G. R., "Digital Computer Subroutine for the Conversion of Oxygen Tension into Saturation", *J. Appl. Physiol.*, 21, (1966), 1375-1376.
2. Murray, J. D., "On the Molecular Mechanism of Facilitated Oxygen Diffusion by Hemoglobin and Myoglobin", *Proc. Soc. Lond. B*, (1971), 178, 95-110.
3. Kapur, J. N., "Mathematical Models in Biology and Medicine", Ad. Wil. East Publication, India, (1985).
4. Sharan, M., Singh, M. P. and Kumar, P., "Finite-Element Analysis of Oxygen Transport in the Systemic Capillaries", *IMA J. Math. Appl. Biol. and Med.*, 8, (1991), 107-123.
5. Salathe, E. P. and Xu, Y. H., "Non-Linear Phenomena in Oxygen Transport to Tissue", *J. Math. Biol.*, 30, (1991), 151-160.
6. Salathe, E. P. and Chen, C., "The Role of Myoglobin in Retarding Oxygen Depletion in Skeletal Muscle", *Math. Bio-Sci.*, (1993), 116, 1-20.
7. Schubert, R. W. and Zhang, Z., "The Equivalent Krogh Cylinder and Oxygen Transport to Tissue", *Journal of Bio. Mech.*, XVIII, (1997), 191-202.
8. Sharma, G. C., Kumar S., Jain, M. and Saral, R., "The Effects of Non-Zero Bulk Flow and Non-Mixing on Diffusion with Variable Transfer of the Solute", *Int. Journ. Appl. Sc. Computation*, 4, No. 3, (1998), 201-210.
9. Whitely, J. P., Gavaghan, D. J. and Hahn, C. E. W., "Mathematical Modeling of Pulmonary Gas Transport", *J. Mathematical Biolog.*, (2003).



**Figure 5.** The solution of the governing equations, Equations 1-6 and 9 with  $P_a = 600$  mmHg.