RESEARCH NOTE

ON THE APPROXIMATION OF PSEUDO LINEAR SYSTEMS BY LINEAR TIME VARYING SYSTEMS

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Abstract This paper presents a modified method for approximating nonlinear systems by a sequence of linear time varying systems. The convergence proof is outlined and the potential of this methodology is discussed. Simulation results are used to show the effectiveness of the proposed method.

Key Words Pseudo Linear Systems, Iterative Method, Time Varying Systems

چکیده در این مقاله روشی اصلاح شده برای تقریب سیستمهای غیرخطی توسط دنباله ای از سیستمهای خطی متغیر با زمان ارائه می‌گردد. همگرایی روش ارائه شده، اثبات گردیده است و با ارائه نتایج شبیه سازی، توانمندی روش به نمایش گذاشته شده است.

1. INTRODUCTION

The theory of nonlinear systems has been an important area of research for many years [2,5,6,10]. Pseudo linear systems are an important class of nonlinear systems. The stability and robustness of these systems are considered in references 2 and 3. The approximation of bilinear systems by a sequence of linear time varying systems has recently attracted much attention [1,8,9]. This paper presents a new method for approximating nonlinear systems by a sequence of linear time varying systems. It is shown that, by updating the initial conditions based on the closeness of the approximation and the exact solution, and dividing the settling time of the system into small intervals, faster convergence is achieved. The convergence proof of the algorithm is also provided and the nonlinear Van Der-Pol example is used to show the effectiveness of the proposed method.

2. STATEMENT OF THE PROBLEM

Consider the pseudo linear systems governed by the following nonlinear differential equation [3]:

$$\dot{x} = A(x)x \tag{1}$$

where A(.): $R^n \rightarrow R^{n^2}$ is a continuously differentiable matrix valued function. A method for approximating the non-linear system given by Equation 1 by a sequence of linear time varying systems is developed in this paper.

Consider the approximating sequence

$$\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t) \qquad (2)$$

with $\dot{x}^{[0]}(t) = A(x_0)x^{[0]}(t)$, $x^{[i]}(0) = x_0$.

It will be shown in the next section that, the solution of equations given in Equation 2, converges to the exact solution of the nonlinear system given by Equation 1. Let

$$x^{[i]}(t) = \phi^{[i-1]}(t,0)\,x^{[i]}(0) \qquad (3)$$

represent this solution, where it converges in $C([0,T];R^n)$ and $C([0,T];R^n)$ denotes the space of continuous functions on [0,T] with values in R^n.

3. PROOF OF CONVERGENCE

It is obvious that from Equation 2, we have

$$\begin{aligned}\dot{x}^{[i]} - \dot{x}^{[i-1]} &= A(x^{[i-1]}(t))x^{[i]}(t) - A(x^{[i-2]}(t))x^{[i-1]}(t)\\ &= A(x^{[i-1]}(t))[x^{[i]}(t) - x^{[i-1]}(t)]\\ &\quad + [A(x^{[i-1]}(t)) - A(x^{[i-2]}(t))]x^{[i-1]}(t)\end{aligned} \qquad (4)$$

consider the following lemma which is used in the development of the proof.

Lemma 3.1 Assume that

$$\| x^{[i]}(0) \| \le \psi$$
$$\| A(x) - A(y) \| \le \alpha \| x - y \|$$
$$\mu(A(x)) \le \mu$$

for all x,y $\in R^n$ and for some $\alpha, \psi, \mu > 0$ where, ||.|| denotes the norm and

$$\mu(A) = \lim_{h \to 0^+} \frac{\| I + hA \| - 1}{h}$$

is the logarithmic norm of A [4].

Proof Let

$$\xi^{[i]}(t) = x^{[i]}(t) - x^{[i-1]}(t)$$

therefore

$$\begin{aligned}\xi^{[i]}(t) &= \phi^{[i-1]}(t,0)\xi^{[i]}(0)\\ &\quad + \int_0^t \phi^{[i-1]}(t,s)[A(x^{[i-1]}(s))\\ &\quad - A(x^{[i-2]}(s))]x^{[i-1]}(s)ds\end{aligned} \qquad (5)$$

where $\phi^{[i-1]}(t,x)$ is the transition matrix generated by $A(x^{[i-1]}(t))$ and

$$\xi^{[i]}(0) = x^{[i]}(0) - x^{[i-1]}(0)$$

It is well known that [4]

$$\| \phi^{[i-1]}(t,s) \| \le \exp[\int_s^t \mu(A(x^{[i-1]}(\tau))d\tau]$$

and so

$$\| \phi^{[i-1]}(t,s) \| \le \exp[\mu(t-s)].$$

Hence taking norm from Equation 5 and by Lemma 3.1 we get

$$\| \xi^{[i]}(t) \| \le e^{\mu t} \| \xi^{[i]}(0) \| + \alpha \ \psi \int_0^t e^{2\mu t - \mu s} \| \xi^{[i-1]}(s) \| ds \qquad (6)$$

Let

$$\eta_i = \sup_{[0,T]} \| \xi^{[i]}(t) \| \qquad t \le T$$

Therefore

$$\eta_i \le e^{\mu T} \| \xi^{[i]}(0) \| + \alpha \psi \, [\int_0^T e^{2\mu t - \mu s} ds]\eta_{i-1}$$

Let

$$e^{\mu T} \| \xi^{[i]}(0) \| = c_i \quad , \quad \int_0^T e^{2\mu t - \mu s} ds = c(T)$$

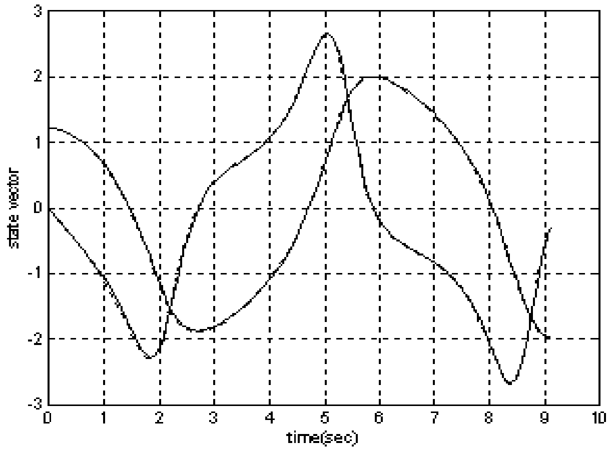

Figure 1. The solid lines show the exact solution and the dots represent the approximate solution of the states of van der pol oscillator with the initial condition $x_0 = [0\ 1.2]^T$.

Then

$$\eta_i \le c_i + \alpha\psi\, c(T)\eta_{i-1}$$
$$\eta_{i-1} \le c_{i-1} + b\eta_{i-2}$$

where

$$b = \alpha\psi\, c(T).$$

Using the recursive method we get

$$\eta_i \le \sum_{k=0}^{i-1} b^k c_{i-k} + b^i\eta_0$$

If we choose c_i such that

$$c_i \le \frac{1}{2^i}$$

then

$$\eta_i \le \sum_{k=0}^{i-1} b^k \frac{1}{2^{i-k}} + b^i\eta_0$$
$$\eta_i \le \frac{1}{2^i}\frac{1}{1-2b} + b^i\eta_0$$

if $b < \frac{1}{2}$ then $\eta_i \to 0$ as $i \to \infty$.

The above results are summarized in the following theorem.

Theorem 3.1 Under the assumptions of Lemma 3.1, if $b < \frac{1}{2}$, then the approximating solution given by 2 converges to the exact solution of the nonlinear system described by 1.

4. ILLUSTRATIVE EXAMPLE

The modified method is applied to approximate the van der pol oscillator given by

$$\dot{x} = A(x)x$$

where

$$A(x) = \begin{bmatrix} 1 - x_2^2 & -1 \\ 1 & 0 \end{bmatrix} \qquad x_0 = [0 \qquad 1.2]^T$$

the simulation results are shown in Figure 1.

5. CONCLUSION

It is shown that the pseudo linear systems can be considered as limits of linear time varying systems. Therefore, the linear techniques can be used for such plants. A numerical example is used to show the speed of convergence of the proposed method.

6. REFERENCES

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