RESEARCH NOTE

SLOPE STABILITY ANALYSIS USING A NON-LINEAR OPTIMIZATION TECHNIQUE

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Abstract In this study, a limit equilibrium method has been developed that satisfies all conditions of equilibrium and assumes circular slip surfaces. All force and moment equilibrium equations are employed without using simplification assumptions. A non-linear optimization technique is used to solve the system of equations with the corresponding constraints. The proposed method is capable to determine the interslice forces, factor of safety, and the coordinates of the critical slip surface center and the length of its radius. Examples for various unknown slope stability parameters are presented and compared to other conventional methods. The concept of the proposed method can be simply extended to multi-layered soil problems with circular and non-circular slip surfaces.

Key Words Slope Stability, Limit Equilibrium, Nonlinear Optimization, Slip Surface, Factor of Safety

چکیده در این تحقیق یک روش تعادل حدی توسعه داده شده است که کلیه شرایط تعادل را ارضا می‌نماید و سطوح لغزش را دایره ای فرض می‌کند. معادلات تعادل نیروها و گشتاورها بدون هیچگونه فرض ساده کننده در این روش بکار برده شده اند. روش بهینه سازی غیر خطی برای حل این معادلات بهمراه قیود مربوطه استفاده شده است. روش ارائه شده توانایی تعیین نیروهای بین قطعه ای، مختصات مرکز دایره لغزش و شعاع آن را دارا می‌باشد. مثالهایی با پارامتر های پایداری سطوح شیبدار مختلف ارائه و نتایج آن با روشهای متداول دیگر مقایسه شده است. مفاهیم روش پیشنهادی به راحتی قابل بسط به خاکهای لایه ای با سطوح لغزش دایره ای و غیر دایره ای می‌باشد.

1. INTRODUCTION

Many methods for analyzing slope stability have been developed. The limit equilibrium methods are considered the most common ones for practical purposes (Duncan [1]). Equilibrium methods, such as Lowe and Karafiath [2] and U.S. Army Corps of Engineers [3] satisfy force equilibrium conditions. Ordinary method of slices (Fellenius [4]) satisfies moment equilibrium conditions. Bishop’s modified method [5] satisfies moment and vertical force equlibriums. Morganstern and Price’s method [6], Janbu’s generalized procedure of slices [7], and Spencer’s method [8] satisfy all conditions of equilibrium. The number of equilibrium equations available is less than the number of unknowns in slope stability problems. Therefore, the problem is indeterminate. All equilibrium methods employ assumptions to render the problem determinate. In the case of methods that satisfy all conditions of equilibrium, it has been found that the error in estimating the factor of safety is much less than the other equilibrium methods. Force equilibrium methods do not afford as high a degree of accuracy

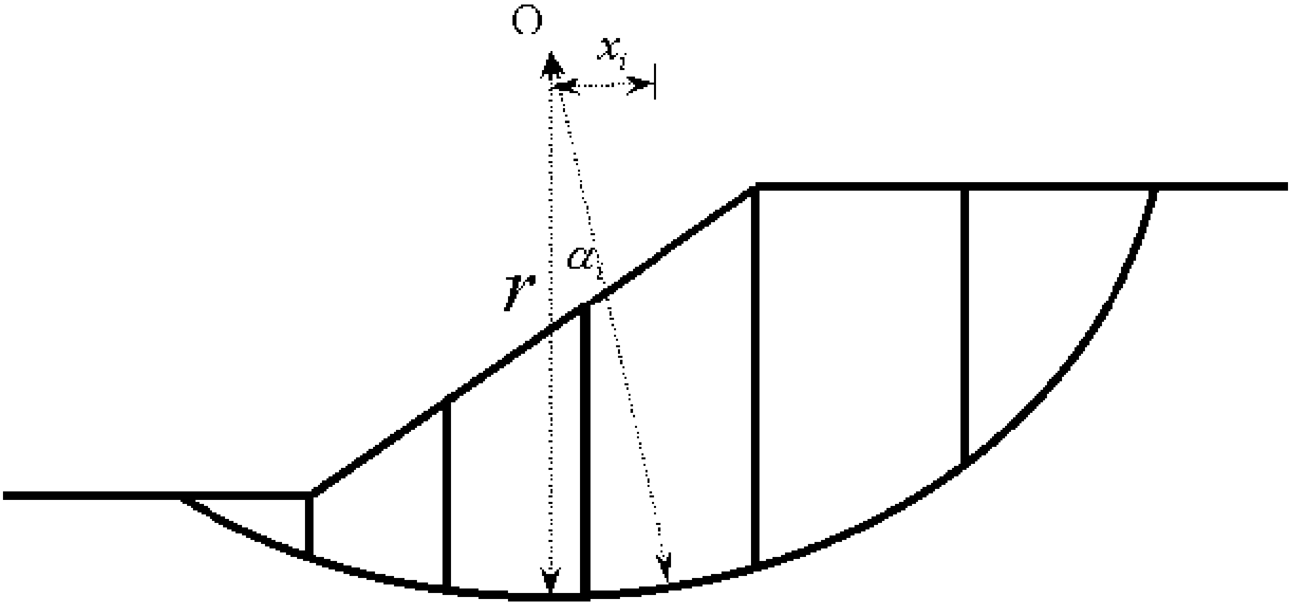

Figure 1. Sliding circular surface subdivided into vertical slices.

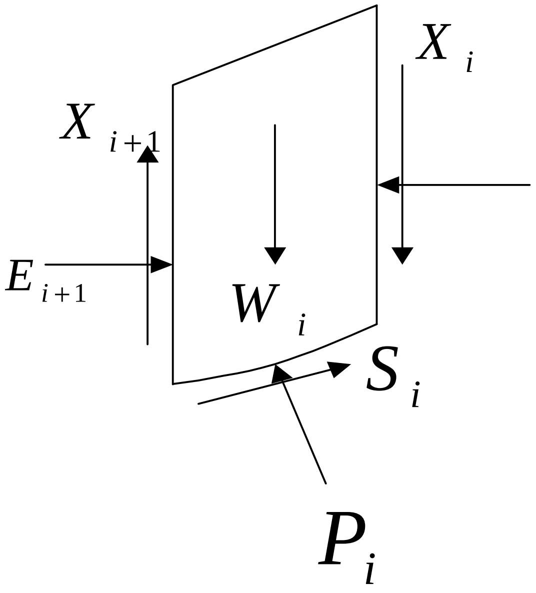

Figure 2. Free body diagram of a slice.

as do methods that satisfy all conditions of equilibrium (Duncan and wright [9]).

Locating the slip surface with the lowest factor of safety is an important part of analyzing slope stability. Most of the methods that assume circular critical slip surfaces use systematic changes in the position of the center of circle and length of the radius to find the critical circle that has the lowest factor of safety. Nguyen [10] and Chen and Shao [11] used optimization techniques to find the critical slip surface. Spencer [12] found that circular slip surfaces were as critical as logarithmic spiral slip surfaces for all practical purposes. Celestino and Duncan [13] and Spencer found that, in analyses where the slip surface was allowed to take any shape, the critical slip surface found by the search was essentially circular.

In this study, a method has been developed that satisfies all conditions of equilibrium and assumes the slip surfaces to be circular. A non-linear optimization technique has been employed in the analysis to determine the unknown parameters.

2. PROPOSED METHOD

Figure 1 shows a potential sliding mass along a trial slip surface through a homogeneous slope. The sliding mass is subdivided into a number of vertical slices. The free body diagram of a slice is illustrated in Figure 2 The forces acting on the slice are its own weight W_i , side forces, both which have shear component X_i , and normal components E_i , and the shear resistance S_i and the normal force P_i which act on the base of the slice. Equating the moment of the weight of the sliding mass with the moment of the external forces acting on the slip surface, about the center O of the slip circular surface yields:

$$\sum W_i \cdot x_i = \sum S_i \cdot r \tag{1}$$

in which X_i and r are shown in Figure 1.

The relation between the shear strength of failure and equilibrium shear stress along the shear surface can be expressed as:

$$\tau = \frac{\tau_f}{F} \tag{2}$$

in which F is the factor of safety. Combining Equation 2 with the Mohr-Columb equation gives:

$$\tau = \frac{1}{F}\left[C' + \left(P_i / l_i - u_i\right) \cdot \tan \phi'\right] \tag{3}$$

where:

C' = drained cohesion of the soil
ϕ' = drained internal friction angle
l_i = the slice base length
u_i = pore water pressure

Vertical equilibrium for the slice i gives:

$$W_i + X_i - X_{i+1} = P_i \cdot \cos\alpha_i + S_i \cdot \sin\alpha_i \tag{4}$$

Resolving for P_i yields:

$$P_i = \left(W_i + X_i - X_{i+1}\right) \cdot \sec\alpha_i - S_i \cdot \tan\alpha_i \tag{5}$$

Substituting the last expression in Equation 3 and after manipulation gives:

$$S_i = \frac{1}{F + \tan\alpha_i \cdot \tan\phi'} \sum \{C' \cdot l_i + \left[\left(W_i + X_i - X_{i+1}\right) \cdot \sec\alpha_i - u_i \cdot l_i\right] \cdot \tan\phi' \} \tag{6}$$

Hence, by substituting the last expression for S_i in Equation 1 yields:

$$r \sum \frac{C' \cdot l_i + \left[\left(W_i + X_i - X_{i+1}\right) \sec\alpha_i - u_i \cdot l_i\right] \tan\phi'}{F + \tan\alpha_i \cdot \tan\phi'} - \sum r \cdot W_i \cdot \sin\alpha_i = 0 \tag{7}$$

The summation of the normal interslice forces should also be zero:

$$\sum \left(E_i - E_{i+1}\right) = 0 \tag{8}$$

Resolving the forces acting on the slice in a tangential direction to the base of the slice:

$$S_i = (E_i - E_{i+1}) \cdot \cos\alpha_i + (W_i + X - X_{i+1}) \cdot \sec\alpha_i \tag{9}$$

Therefore:

$$\sum (E_i - E_{i+1}) = \sum S_i \cdot \sec\alpha_i - (W_i + X_i - X_{i+1}) \cdot \tan\alpha_i \tag{10}$$

Insertion of the value of S_i from Equation 6 into Equation 10 yields:

$$\sum \frac{C' l_i + \{\left(W_i + X_i - X_{i+1}\right) \sec\alpha_i - u_i l_i\} \tan\phi'}{F + \tan\alpha_i \cdot \tan\phi'} \sec\alpha_i - \left(W_i + X_i - X_{i+1}\right) \tan\alpha_i = 0 \tag{11}$$

Equations 7 and 11 are the moment and force equilibrium equations, respectively. These equations are considered to be the fundamental equations, which should be solved to determine the unknowns X_i for every slice and the factor of safety F. The number of equations is less than the number of unknowns and the system is thus indeterminate.

3. OPTIMIZATION TECHNIQUE

In this study, it is desired to solve Equations 7 and 11 to obtain the shear interslice forces X_i, and the factor of safety F. It is obvious that this system of equations has infinite number of solutions, but it is possible to constraint the unknown values to lower and upper limits in order to be able to obtain the appropriate solution. This requires adequate experience and engineering judgment. In this study, an optimization technique has been used to solve the problem. In this regard the objective function is selected as:

$$\text{Objective Function} = (\text{Eq.}7)^2 + (\text{Eq.}11)^2 \qquad (12)$$

which is subjected to the following constraints:

$$\begin{aligned}
(\text{Lower Limit})_{X_1} &\le X_1 \le (\text{Upper Limit})_{X_1} \\
(\text{Lower Limit})_{X_2} &\le X_2 \le (\text{Upper Limit})_{X_2} \\
&\vdots \\
(\text{Lower Limit})_{X_N} &\le X_N \le (\text{Upper Limit})_{X_N} \\
(\text{Lower Limit})_{X_c} &\le x_c \le (\text{Upper Limit})_{X_c} \\
(\text{Lower Limit})_{Y_c} &\le y_c \le (\text{Upper Limit})_{Y_c} \\
(\text{Lower Limit})_{r} &\le r \le (\text{Upper Limit})_{r} \\
0 &\le F \le (\text{Upper Limit})_{F}
\end{aligned} \qquad (13)$$

The transformed conjugate nonlinear optimization method (Box 1966) is used for minimizing the objective function given by Equation 12. In this technique which is an iterative method, each iteration of the procedure commences with a search down n linearly independent directions called the conjugate directions. The method does not require calculation of derivatives and it is based on a searching procedure. The method starts searching from different initial points distributed in the problem domain in order to find the global minimum. It is obvious that the unknowns which satisfy Equations 7 and 11 and the constraints given by Equation 13 should make the objective function given by Equation 12 equal or close to zero. This goal is achieved by the optimization technique.

A computer program has been developed in this study in which circular slip surfaces are concerned and the slip surface is divided into a number of vertical slices. Based on the dimensions of the slice and soil properties, the interslice forces are determined. This program is linked to the optimization program in order to determine the unknowns given by Equation 12 and 13. The coordinates of the critical slip surface and its center are obtained by a searching technique in a given domain . The optimizatin technique solves for the factor of safety and interslices forces starting from an arbirary circle and proceeds for other circles in the given domain. The critical slip circle will be the one with the lowest factor of safty. The computer program is adapted by such a way that if desired any of the unknowns can be taken out of the optimization procedure and given as known parameters.

In this stage of the study, to examine the method, a number of simple cases with homogeneous and dry soils have been analyzed.

4. ILLUSTRATIVE EXAMPLES

Example 1 The geometry of the slope is shown in Figure 3. The parameters of the soil are given as:

γ_d = 16 kN/m^3
c' = 10 kPa
ϕ' = 15°

The coordinates of the slip circle center and its radius are:

x_c = 12.6 m.
y_c = 20.6 m.
r = 10.6 m.

As was mentioned before, a computer program has been written so that any of the unknowns can be taken out of the optimization procedure and given as known parameters. In this example, the slip surface geometry parameters are taken out of the optimization procedure and given as known parameters. In examples 3 and 4, which will be illustrated later, the slip circle geometry parameters are considered to be unknowns, i.e., they are included in the optimization parameters.

Factors of safety determined by the proposed method and other conventional slope stability analysis methods for 10 slices are given in Table 1. Interslice shear forces are given in Table 2.

Example 2 The geometry of the slope is depicted

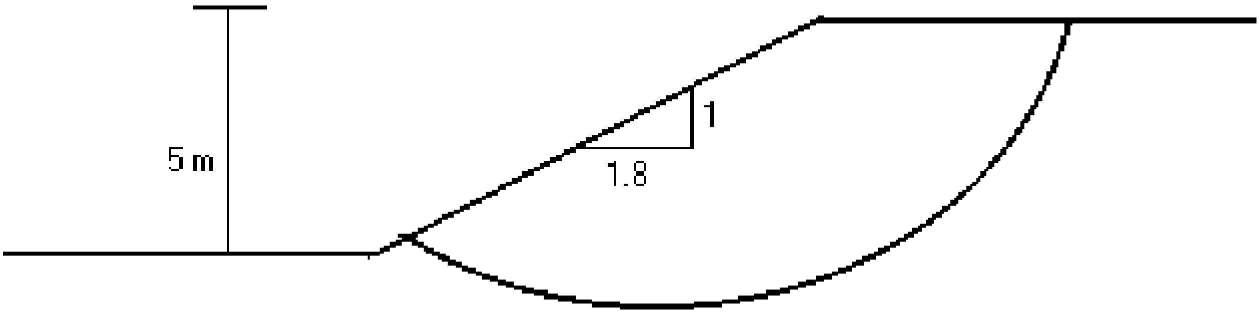

Figure 3. Geometry of slope of Example1.

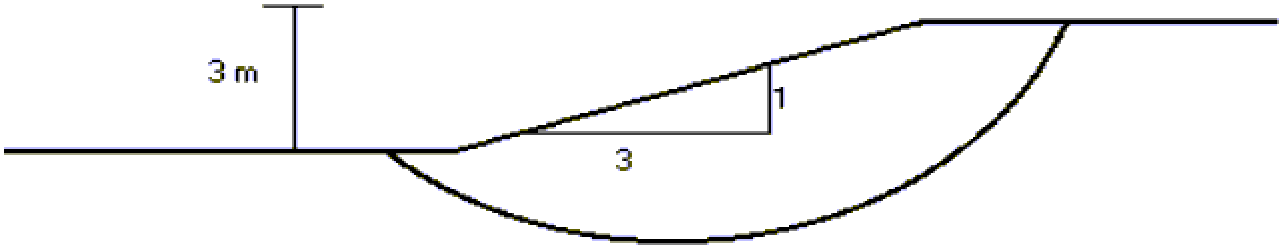

Figure 4. Geometry of slope of Example 2.

TABLE 1. Factors of Safety Calculated by Different Methods for Example 1.

Fellenius	Bishop	Janbu	Morgenstern-Price	Proposed method
1.634	1.688	1.588	1.686	1.700

in Figure 4. The soil parameters are given as:

γ_d = 18 kN/m^3

c' = 12 kPa

ϕ' = 10°

The coordinates of the slip circle center and its radius are:

x_c = 14.2 m.

y_c = 17.4 m.

r = 9.4 m.

Results of this example are given in Tables 3 and 4.

Example 3 The slope of this example is illustrated in Figure 5. In this example, it is desired to find the coordinates of the critical slip circle center and its diameter in addition to the shear interslice forces and factor of safety.
Soil properties are given as:

γ_d = 16 kN/m^3

TABLE 2. Interslice Shear Forces Calculated by Different Methods for Example 1.

Constant Interslice	Half-Sine	Corps of	Lowe-Karafiath	Proposed
3.81	1.62	0.96	2.14	0.60
7.15	5.31	3.35	4.98	8.60
9.86	9.75	6.76	8.2	11.7
11.41	12.92	10.27	10.95	13.6
11.46	13.35	12.64	12.37	13.9
9.85	10.76	12.7	11.7	12.35
7.57	7.39	10.72	9.55	11.8
3.32	2.46	5.21	2.54	1.69
-0.4	-0.2	-0.58	-0.35	-1.35

TABLE 3. Factors of Safety Calculated by Various Methods for Example 2.

Fellenius	Bishop	Janbu	Morgenstern-Price	Proposed method
2.263	2.394	2.125	2.391	2.437

TABLE 4. Interslice Shear Forces Calculated by Various Methods for Example 2.

Constant	Half-Sine	Corps of	Lowe-Karafiath	Proposed
3.38	1.39	3.4	-3.21	3.0
7.27	5.49	7.3	0.35	7.4
10.95	11.29	11.0	4.41	12.3
13.54	16.28	13.6	10.5	19.7
14.44	18.1	14.5	15.26	21.8
14.0	17.04	14.06	19.8	22.5
10.7	10.92	10.74	19.33	22.1
7.35	6.12	7.4	9.44	21.5
2.44	1.24	2.45	3.8	12.

$c' = 10$ kPa

$\phi' = 10°$

The results found by the proposed method and other methods are given in Tables 5 and 6. The calculated coordinates of the of critical slip circle

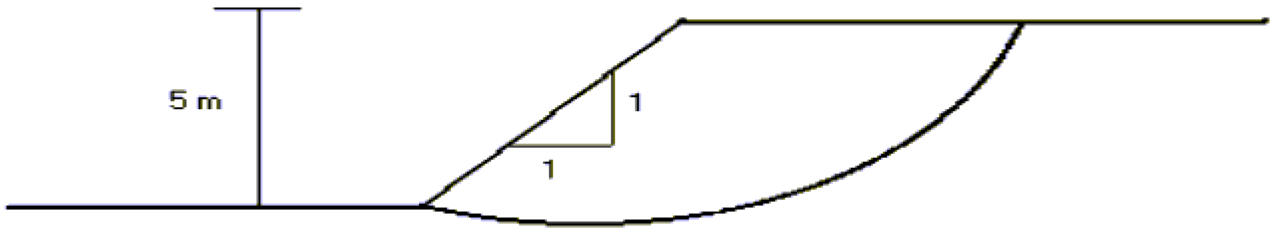

Figure 5. Geometry of slope of Example 3.

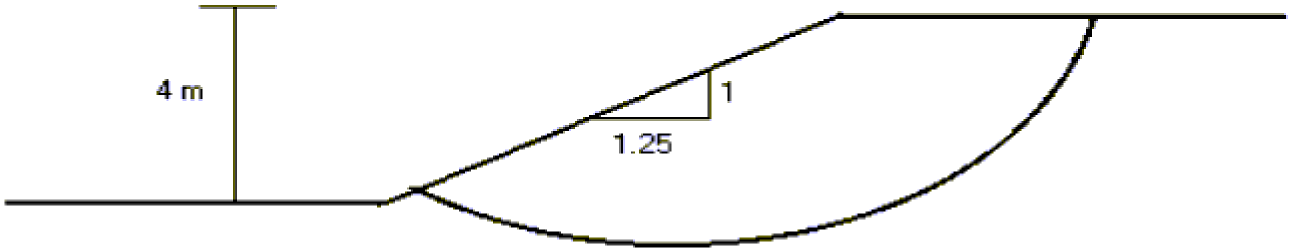

Figure 6. Geometry of slope of Example 4.

center and radius are:

$x_c = 10.8$ m.

$y_c = 17.0$ m

$r = 7.0$ m.

Example 4 Geometry of the slope is shown in Figure 6. Similar to the previous example, x_c, y_c and r of the critical slip circle in addition to the interslice shear forces and factor of safety are to be determined. Soil properties are given as:

$\gamma_d = 16 \text{kN/m}^3$

$c' = 10$kPa

$\phi' = 10°$

The results associated with the factor of safety and the shear interslice forces are given in Tables 7 and 8. The calculated coordinates of the of critical slip circle center and radius are:

$x_c = 10.8$ m.

$y_c = 16.2$ m

$r = 6.2$ m.

5. CONCLUSION

In general, the results found by the proposed method indicate that it gives slightly higher factor of safeties. The main advantage of the developed method compared to other methods that satisfy both force and moment equilibriums is that no simplification assumptions are required to be used

TABLE 5. Factors of Safety Calculated by Various Methods for Example 3.

Fellenius	Bishop	Janbu	Morgenstern-Price	Proposed method
1.081	1.103	1.069	1.102	1.106

TABLE 6. Interslice Shear Forces Calculated by Various Methods for Example 3.

Constant interslice	Half-sine	Corps of	Lowe-Karafiath	Proposed method
3.81	0.27	1.48	0.68	0.46
6.96	0.87	2.75	1.4	5.5
9.33	1.6	3.73	2.1	8.6
10.42	2.08	4.21	2.56	9.08
9.9	2.05	4.02	2.62	11.23
7.5	1.42	3.01	2.07	11.54
3.3	0.41	1.17	0.31	11.5
-1.01	-0.3	-0.78	-0.32	10.9
-3.26	-0.36	-1.82	-0.77	5.66

in the developed method. The non-linear optimization technique employed in this study to solve the system of the equilibrium equations with the corresponding constraints is a powerful mean because it does not require calculation of derivatives and it is based on a searching procedure. In this study, simple cases are concerned in which the soil is homogeneous, dry and there is no earthquake force acting on the slices. The concept of the proposed method can be simply extended to multi-layered soil problems with circular and non-circular slip surfaces.

6. ACKNOWLEDGEMENT

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7. APPENDIX I NOTATION

Xi	interslice shear force component
Ei	interslice normal force component
S_i	shear resistance on the base of the slice
	shear stress of faillure
Pi	normal force on the base of the slice
Wi	slice weight
F	factor of safety
τ_f	shear strength of failure
τ	equilibrium shear stress
C'	cohesion of the soil

TABLE 7. Factors of Safety Calculated by Various Methods for Example 4.

Fellenius	Bishop	Janbu	Morgenstern-Price	Proposed method
1.401	1.428	1.366	1.426	1.450

TABLE 8. Interslice Shear Forces Calculated by Various Methods for Example 4.

Constant Interslice Force	Half-Sine	Corps of Engineers	Lowe-Karafiath	Proposed Method
1.77	0.58	1.75	1.13	0.23
3.10	1.76	3.07	2.26	5.16
4.08	3.04	4.04	3.32	5.77
4.52	3.83	4.5	4.04	7.4
4.4	3.82	4.37	4.2	8.6
3.6	2.96	3.57	3.67	8.3
2.04	1.4	2.02	2.18	8.3
-0.04	-0.11	-0.04	-0.06	7.7
-1.42	-0.5	-1.41	-0.99	-1.07

ϕ'	internal friction angle
l_I	the slice base length
u_I	pore water pressure
α_i	angle between vertical line that passes through the circle center and the radius that passes through the middle of the slice base
x_c	x coordinate of slip circle
y_c	y coordinate of slip circle
r	radius of slip circle
γ_d	dry unit weight

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