DIFFUSION PROCESS FOR $G^X/G/m$ QUEUEING SYSTEM WITH BALKING AND RENEGING

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(Received: December 10, 2001 – Accepted in Revised Form: December 12, 2002)

Abstract In the present investigation transient, $G^X/G/m$ queuing model with balking and reneging has been studied. The diffusion process with elementary return boundary has been used for modeling purpose. The probability density function (p. d. f.) for the number of customers in the system has been obtained. In special case, the steady state results that tally with those of Kimura and Ohsone have been established as a limiting case.

Key Words Diffusion Process, Balking, Reneging, Transient, Elementary Return Boundary, Probability Density Function

چکیده در تحقیق حاضر مدل صف گذاری $G^X/G/m$ با ورودی دسته ای و امتناع کننده از پیوستن به صف مورد مطالعه قرار گرفته است. برای مدل کردن این سیستم از فرآیند پراکنش با شرایط مرزی برگشت استفاده شده است. تابع چگالی احتمال (p.d.f.) تعداد مشتری در سیستم به دست آورد شده است. در حالت خاص نتایج حالت پایدار این مسئله با نتایج Kimura و Ohsone در سال ۱۹۸۴ در حالت حدی تطبیق داده شده است.

1. INTRODUCTION

Although multi-server bulk queuing systems with general service time distribution have a number of applications to various practical problems e. g. inventory, production processes, data transmission systems, it is extremely difficult to investigate transient behavior of such systems analytically. The approximation can play a major role in understanding the complex queuing systems for which exact analytical results are not known. During last two decades diffusion approximation has received considerable attention in literature to analyze general queuing models in different frame-works. In real life, we come across the situations where it is essential to know the transient behavior of the queuing models. Duda [1] studied the transient diffusion approximation with elementary return boundary (E. R. B) for single server. Sivazlian and Wang [2] developed diffusion approximation to the G/G/R machine repair problem with warm standby spares. Choi and Shin [3] gave the transient diffusion approximation for M/G/m system. Jain [4] derived some performance measures for a $G^X/G/m$ machine interference problem with spare machines by using diffusion process with reflecting boundaries. Di Crescenzo et al. [5] discussed diffusion approximation for finite calling population model based on two boundary policies; the elementary return boundary and the instantaneous return boundary. Jain [6] developed (m, M) machine repair problem with spares and state dependent rates by using diffusion approximation technique with reflecting boundaries. Recently, Jain [7] considered the $G^X/G^Y/1$ double-ended queue by using diffusion process. Jain and Singh [8] gave diffusion process for optimal flow control of a G/G/c finite capacity queue. Jain et al. [9] analyzed $G^X/G^Y/R$ machine repair system with group failure, group repair and additional repairmen using diffusion approximation approach.

Due to real life applications in many congestion situations, queuing models with balking and reneging have attracted the researchers working in the field of queuing theory. Some research works have been reported on queuing systems with balking and reneging using diffusion process. Varshney et al. [10] suggested diffusion approximation for G/G/m queue with discouragement. Garg et al. [11] provided approximate solution based on diffusion process with reflecting boundary for $G^X/G/m$ queue with discouragement. Jain [12] developed G/G/1 double-ended queue with balking by using diffusion approximation technique. Jain et al. [13] developed the diffusion process for multi-repairmen machining system with spares and balking.

This paper studies $G^X/G/m$ queuing system with balking and reneging. The diffusion process has been established by using elementary return boundary in Section 2. The transient diffusion approximation for the number of customers in the system has been obtained in Section 3. In particular, for M/G/1 queue, our model reduces to Choi and Shin [3] model. In Section 4, we discuss the steady state results, which tally for special case with those of Kimura and Ohsone [14]. Finally, the concluding remarks and noble features of the investigation are outlined in Section 5.

2. THE MODEL DESCRIPTION AND DIFFUSION PROCESS

We analyze $G^X/G/m$ queuing system with balking and reneging in which customers arrive in batches with random sizes at the service facility with a distribution $\{g_k\}$, k = 1, 2, ... Let $\overline{X}$ and σ_X^2 denote the mean and variance of batch size distribution. The customers are assumed to arrive in general fashion with mean rate λ and square coefficient of variance C_a^2. When all servers are busy, the customers may balk with probability 1-β. The customers are served by one of the m servers in order to their arrivals i. e. in FIFO order. The service times of the customers are independent and identically distributed (i. i. d.) random variables with mean rate μ and finite square coefficient of variance C_s^2. The customers while waiting in the queue may renege exponentially with parameter ν. We assume that Q (t) denotes the number of customers in the system at time t (≥ 0).

We consider diffusion process $\{X(t):\ t\ \geq 0\}$ which approximates the process $\{Q(t):\ t\geq 0\}$. The process X(.) is represented by two diffusion parameters b(x) and a(x) called infinitesimal mean and variance respectively, and are defined as

$$b(x) = \lim_{\Delta t \to 0} \frac{E[X(t+\Delta t) - X(t) / X(t) = x]}{\Delta t}$$

$$a(x) = \lim_{\Delta t \to 0} \frac{Var[X(t+\Delta t) - X(t) / X(t) = x]}{\Delta t}$$

For $G^X/G/m$ model with balking and reneging, we propose mean and variance of the diffusion process as follows:

i. $0 \leq x \leq m$

$b(x) = \lambda \overline{X} - x\mu$

$a(x) = \lambda (C_a^2 \overline{X}^2 + \sigma_X^2) + x\mu C_s^2$

ii. $x \geq m$

$b(x) = \lambda\beta \overline{X} - [m\mu + (x-m)\nu]$

$a(x) = \lambda \beta(C_a^2 \overline{X}^2 + \sigma_X^2) + [m\mu + (x-m)\nu] C_s^2$

Since the underlying process Q(.) cannot take a negative value, some impenetrable boundary should be placed at the origin of X(.).

It is well known that E. R. B. at the origin is more suitable for queuing systems with batch arrivals; we adopt it for our model. The trajectory of $\{X(t):\ t\ \geq 0\}$with the state space $[0, \infty]$ having an E. R. B. behaves in such a way that when it reaches the boundary at x = 0, it remains there for a random interval of time (i.e. sojourn time). The sojourn time at the origin is taken to be exponentially distributed. Thereafter, the trajectory jumps into the interior of the region and starts from scratch. The time intervals during which the system is empty are also exponentially distributed and the path Q (.) jumps from the origin to Q (.) = k with probability g_k according to the arrival of a new k- size group (k = 1, 2,...).

The diffusion process $\{X(t): t \geq 0\}$ with the E. R. B. satisfies the following differential equations (see Feller [15])

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\{a(x)\ f(x,t/x_0)\} - \frac{\partial}{\partial x}\{b(x)\ f(x,t/x_0)\} + \lambda p(t)\sum_{k=1}^{\infty} g_k\,\delta(x-k) \tag{1}$$

and

$$\frac{\partial P(t)}{\partial t} = -\lambda p(t) + \frac{1}{2}\frac{\partial}{\partial x}\{a(x)\ f(x,t/x_0)\} - b(x)\ f(x,t/x_0)\Big|_{x\uparrow 0} \tag{2}$$

where $f(x, t/x_0)$ denotes the probability density function of X (t), P (t) is the probability that the process X (t) is at the origin at time t and δ (.) is the Dirac delta function. The conservation of probability leads to the following conditions:

$$P(t) + \int_0^{\infty} f(x,t/x_0)\,dx = 1 \tag{3}$$

The boundary condition at x = 0 and the initial condition at t = 0, are given by

$$\lim_{x\to 0} f(x,t/x_0) = 0 \tag{4}$$

$$f(x, 0/x_0) = \delta(x-x_0) \tag{5a}$$

$$P(0) = \begin{cases} 0 & \text{if } x_0 > 0 \\ 1 & \text{if } x_0 = 0 \end{cases} \tag{5b}$$

It is evident that there exits a continuous solution of Equation 1 even if the functions b(x) and a(x) are piecewise continuous with a finite number of first order discontinuities.

For k = 1, 2,..., we introduce the following notations:

$$a_k \equiv a(k)$$
$$b_k \equiv b(k)$$

and

$$\bar{g}_k = p(X \geq k) = 1 - \sum_{i=1}^{k-1} g_i$$

Because of the continuity of $f(x, t/x_0)$, we impose the following smoothing conditions:

$$\lim_{\substack{x\downarrow k-1 \\ k=2,3,....}} f_k(x,t/x_0) = f_{k-1}(k-1,t/x_0) \tag{6}$$

For notational convenience, we denote

$q_k(t/x_0) = f_k(k, t/x_0)$
k = 1, 2, ..., m-1.

and

$$f_k(k-1,t/x_0) = \lim_{x\downarrow k-1} f_k(x,t/x_0)$$
$$k = 1,\ 2,\ ......,\ m$$

From the condition (4), we have $q_0(t/x_0) = 0$ for all $t \geq 0$. Now the problem of solving the differential Equation 1 reduces to the following initial boundary value problem,

$$\frac{\partial f_k}{\partial t} = \frac{1}{2}a_k\frac{\partial^2 f_k}{\partial x^2} - b_k\frac{\partial f_k}{\partial x} + \lambda\ P(t)\sum_{k=1}^{\infty} g_k\,\delta(k-k) \tag{7}$$

$$f_k(k-1,t/x_0) = q_{k-1}(t/x_0) \tag{8a}$$

$$f_k(k,t/x_0) = q_k(t/x_0) \tag{8b}$$

$$f_k(k,0/x_0)=\delta(x-x_0) \tag{8c}$$

for k –1< x < k, t >0, k = 1, 2, ….. , m-1.

3. TRANSIENT SOLUTION

We rewrite the above system of Equations 7 – 8 (a-c) as

$$\frac{\partial f}{\partial t}=\frac{a}{2}\frac{\partial^2 f}{\partial x^2}-b\frac{\partial f}{\partial x}+H \tag{9}$$

$$f(x,0)=\delta(x-x_0) \tag{10a}$$

$$f(k-1,t)=q_{k-1}(t) \tag{10b}$$

$$f(k,t)=q_{k-1}(t) \tag{10c}$$

$$H(x,t)=\lambda\, P(t)\sum_{k=1}^{\infty} g_k\,\delta(x-k) \tag{10d}$$

Let us assume that

$$W(y,t)=F(y,t)\exp\left(-\frac{b}{a}y+\frac{b^2}{2a}t\right)$$

$$H_1(y,t)=H(y,t)\exp\left(-\frac{b}{a}y+\frac{b^2}{2a}t\right)$$

where y = x-k+1 and F(y, t) = f (y + k-1), $0\le y\le 1$ and $t\ge 0$. Then from Equations 8 and 9, we have

$$\frac{\partial W}{\partial t}=\frac{a}{2}\frac{\partial^2 W}{\partial y^2}+H_1\ ;\quad (0<y<1),\quad t>0 \tag{11}$$

$$W(y,0)=\delta(y+k-1-x_0)\exp\left(-\frac{b}{a}y\right)\ ;\ 0<y<1 \tag{12a}$$

$$W(0,\ t)=q_{k-1}(t)\ \exp\left(\frac{b^2}{2a}t\right)\quad ;\quad t\ge 0 \tag{12b}$$

$$W(I_2-I_1,t)=q_k(t)\exp\left(-\frac{b}{a}+\frac{b^2}{2a}t\right)$$
$$t\ge 0 \tag{12c}$$

Solving Equation 11 by taking Laplace transform with respect to t of both sides, we have (see Carrier and Carl [16])

$$\begin{aligned}W^*(y,s)=&\\ &\frac{\text{Sinh}\sqrt{\frac{2s}{a}}}{\text{Sinh}\sqrt{\frac{2s}{a}}(I_2-I_1)}\Bigg[q_k^*\left(s-\frac{b^2}{2a}\right)\exp\left(-\frac{b}{a}(I_2-I_1-1)\right)\\ &-q_k^*\left(s-\frac{b^2}{2a}\right)\text{Cosh}\left(\sqrt{\frac{2s}{a}}(I_2-I_1)\right)\\ &+\sqrt{\frac{2}{as}}\int_0^{I_2-I_1}\Big\{\delta(\xi+I_1-x_0)\exp\left(-\frac{b}{a}\xi\right)\\ &+H_1^*(\xi,s)\Big\}\text{Sinh}\left(\sqrt{\frac{2s}{a}}(I_2-I_1-\xi)\right)d\xi\Bigg]\\ &+q_{k-1}^*\left(s-\frac{b^2}{2a}\right)\text{Cosh}\left(\sqrt{\frac{2s}{a}}y\right)\\ &-\sqrt{\frac{2}{as}}\int_0^{y}\Big\{\delta(\xi+I_1-x_0)\exp\left(-\frac{b}{a}\xi\right)+\\ &H_1^*(\xi,s)\Big\}\text{Sinh}\left(\sqrt{\frac{2s}{a}}(y-\xi)\right)d\xi\end{aligned} \tag{13}$$

Simplifying Equation 13 in the same manner as done by Choi and Shin [3], we have

$$f_k^*(x,s/x_0)=$$
$$\exp\left(\frac{b_k(x-k)}{a_k}\right)\frac{\text{Sinh}A_k(x-k+1)}{\text{Sinh}A_k}q_k^*(s/x_0)$$
$$+\exp\left(\frac{b_k(x-k+1)}{a_k}\right)\frac{\text{Sinh}\,A_k(k-x)}{\text{Sinh}\,A_k}q_{k-1}^*(s/x_0)$$

for $k\text{-}1\le x\le k$, $k = 1, 2, \ldots..$ (14)

where

$$A_k = \frac{2a_k s + b_k^2}{a_k}, \qquad k = 1, 2, \ldots\ldots.. \tag{15}$$

Now we will determine $q_k^*(s/x_0)$ in the above expression in terms of known parameters. Taking Laplace transform of Equation 2 with respect to t and then integrating with respect to x, we get

$$\frac{1}{2}\frac{\partial}{\partial x}\left\{a(x)f^*(x,s/x_0)\right\}-b\,(x)f^*(x,s/x_0)$$
$$=\left[V_{x,s}f^*\right]_{x\downarrow 0}+$$
$$s\int_0^x f^*(y,s/x_0)dy-U(x-x_0)-\lambda P^*(s)\sum_{k=1}^{\infty}g_k\,U(x-k) \tag{16}$$

where

$$V_{x,s}f^*=$$
$$\frac{1}{2}\frac{\partial}{\partial x}\left\{a(x)f^*(x,s/x_0)\right\}-b(x)f^*(x,s/x_0)$$

Making use of Equation 14, we have

$$\left[V_{x,s}f_1^*\right]_{x\downarrow k-1}=$$
$$\left[V_{x,s}f_{k-1}^*\right]_{x\uparrow k-1}+\lambda P^*(s)\,[\overline{g}_k-\overline{g}_{k-1}]-I\,(x_0=k-1)$$

for k=1 (17)

Taking Laplace transform of Equation 2, we get

$$\left[V_{x,s}f_1^*\right]_{x\downarrow 0}=(s+\lambda)P^*(s)-P(0) \tag{18}$$

From Equations 14 and 17, we have

$$f_1^*(s)=\frac{1}{B_1}(s+\lambda)P^*(s)\overline{g}_1-\frac{1}{B_1} \tag{19}$$

Similarly from Equations 14 and 18, we get

$$f_k^*(s)=\frac{C_k}{B_k}f_{k-1}^*(s)-\frac{B_{k-1}}{B_k}\exp\left(\frac{2b_{k-1}}{a_{k-1}}\right)f_{k-2}^*(s)$$
$$+\frac{\lambda}{B_k}(\overline{g}_k-\overline{g}_{k-1})P^*(s) \tag{20}$$

where

$$B_k=\frac{a_kA_k}{2}\exp\left(-\frac{b_k}{a_k}\right)\frac{1}{\text{Sinh}\;A_k} \tag{21}$$
$$k=1,2,\ldots..$$

and

$$C_k=-\frac{b_{k-1}}{2}+\frac{a_{k-1}A_{k-1}}{2}\frac{\text{Cosh}\;A_{k-1}}{\text{Sinh}\;A_{k-1}}+$$
$$\frac{b_k}{2}+a_kA_k\frac{\text{Cosh}\;A_k}{\text{Sinh}\;A_k}, \qquad (k=1,2,\ldots.) \tag{22}$$

The normalizing condition (3) leads to

$$P^*(s)+\int_0^{\infty}f^*(x,s/x_0)\;dx=\frac{1}{s}, \qquad s>0 \tag{23}$$

Using Equation 14, $P^*(s)$ can be determined in terms of known parameters.

4. STEADY STATE DIFFUSION APPROXIMATION

Kimura and Ohsone [14] have considered diffusion approximation for $M^X/G/m$ queuing system in the steady state. Now we derive the steady state results for $G^X/G/m$ queuing system with balking and reneging by letting $t \to 0$ in the transient solution obtained in Section 3.

We shall show that in particular case our results are similar to those of Kimura and Ohsone [14]. For steady state when $t \to \infty$, we define

$$\lim_{t \to \infty} f_k(x, t/x_0) = f_k(x)$$

$$\lim_{t \to \infty} g_k(t/x_0) = g_k$$

and

$$\lim_{t \to \infty} P(t) = P$$

From Equations 15, 21 and 22, we have

$$\lim_{s \to 0} A_k(s) = \frac{|b_k|}{a_k}, \quad k = 1, 2, ... \tag{24}$$

$$\lim_{s \to 0} B_k(s) = \frac{b_k}{\exp\left(\frac{2b_k}{a_k}\right) - 1}, \quad k = 1, 2, ... \tag{25}$$

and

$$\lim_{s \to 0} C_k(s) = \frac{b_{k-1}}{\exp\left(\frac{2b_{k-1}}{a_{k-1}}\right) - 1} + \frac{b_k \exp\left(\frac{2b_k}{a_k}\right)}{\exp\left(\frac{2b_k}{a_k}\right) - 1},$$

$$k = 1, 2, ... \tag{26}$$

Using Equation 21, the results become

$$q_k = \lim_{s \to 0} s q_k^*(s) =$$

$$\frac{b_{k-1}}{b_k} \frac{\exp\left(\frac{2b_k}{a_k}\right) - 1}{\exp\left(\frac{2b_{k-1}}{a_{k-1}}\right) - 1} \exp\left(\frac{2b_{k-1}}{a_{k-1}}\right) q_{k-2} -$$

$$\frac{2b_k}{a_k}\left(\exp\left(\frac{2b_k}{a_k}\right) - 1\right)\left(\bar{g}_k - \bar{g}_{k-1}\right)P$$

if $b_k \neq 0$

$$= \exp\left(\frac{2b_k}{a_k}\right) q_k + \frac{\lambda}{b_k}\left(\exp\left(\frac{2b_k}{a_k}\right) - \bar{g}_k\right)$$

if $b_k = 0$

(27)

Now using Equation 14, we get

$$f_k(x) \lim_{s \to 0} s f_k^*(x, s/x_0) =$$

$$\left(q_k + \frac{\lambda}{b_k}\right) \exp\left(\frac{2b_k(x-k)}{a_k}\right) -$$

$$\frac{\lambda P}{b_k} \bar{g}_k, \quad \text{if } b_k \neq 0 \tag{28}$$

From Equation 3, the steady state probability P is given by

$$P = \left[1 + \sum_{k=1}^{m-1}\left[\frac{a_k}{2b_k} - \frac{a_{k+1}}{2b_{k+1}}\right] q_k - \sum_{k=1}^{\infty} \frac{\lambda}{b_k} \bar{g}_k\right] \tag{29}$$

In case when $b_k = 0$ for some k (it should be noted that it is unique if it exists), P can be obtained by letting b_k tends to zero in Equation 29.

5. DISCUSSION

In this paper, we have analyzed a multi-server transient queuing system with group arrivals via diffusion approximation. The considered model is of interest from the view of its practical applications in real life. Using Laplace transform, the p. d. f. for the number of customers in the system is derived. Finally by letting $t \to \infty$ in the transient solution, results for the steady state are obtained.

For discretizing the continuous probability density function f (x, t/x_0), we can adopt any one of the following methods:

(i) $P_n(t) = f(x, t/x_0)$

(ii) $P_n(t) = \int_{n-1}^{n} f(x, t/x_0)dx$

(iii) $P_n(t) = \int_{n}^{n+1} f(x, t/x_0)dx$

(iv) $P_n(t) = \int_{n-0.5}^{n+0.5} f(x, t/x_0)dx$

But it is more convenient to choose method (i). To obtain approximate numerical inversion f(t) of $f^*(s)$ at time t, we can use Stefest's [17] result given by

$$f(t) = \frac{\ln 2}{t} \sum_{i=1}^{N} V_i f^*\left(\frac{\ln 2}{t} i\right)$$

where

$$V_i = (-1)^{N/2+i} \sum_{k=\frac{i+1}{2}}^{\mathrm{Min}(i,N/2)} \frac{k^{N/2}(2k)!}{(N/2-k)!k!(k-1)!(2k-1)!}$$

The novel feature of suggested diffusion approximation lies in the fact that explicit results can be obtained for relatively complex situation in terms of means and variances of inter-arrival service time and group size distributions.

6. ACKNOWLEDGEMENTS

University Grant Commission, New Delhi, vide Project No. F 8 -5/98 (SR-I) has supported this research.

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