TECHNICAL NOTE

DYNAMIC MULTI-PERIOD PRODUCTION PLANNING PROBLEM WITH SEMI-MARKOVIAN VARIABLE COST

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Abstract This paper develops a method for solving the single product multi-period production-planning problem, in which the production and the inventory costs of each period are concave and backlogging is not permitted. It is also assumed that the unit variable cost of the production evolves according to a continuous time Markov process. We prove that this production-planning problem can be stated as a problem of finding the dynamic shortest path from the source node to the sink node. Finally, we apply the stochastic dynamic programming to find the dynamic shortest path from the source node to the sink node and obtain the optimal production scheduled for each period.

Key Words Multi-period Production planning, Shortest Path, Stochastic Dynamic Programming

چکیده این مقاله به توسعه مدلی جهت حل مساله برنامه ریزی تولید چند پریودی تک محصولی، به نحوی که هزینه تولید هر پریود خطی و هزینه نگهداری موجودی در هر پریود یک تابع مقعر باشد و همچنین کمبود مجاز نباشد می‌پردازد. همچنین فرض می‌شود تغییرات هزینه متغیر تولید هر واحد محصول در گذر زمان از یک فرآیند مارکوفی زمان پیوسته تبعیت کند. در ابتدا ثابت می‌کنیم مساله برنامه ریزی تولید فوق را می توان در قالب یک مساله تعیین کوتاهترین مسیر پویا از گره ابتدایی به گره انتهایی در نظر گرفت. نهایتاً از برنامه ریزی پویای تصادفی برای یافتن کوتاهترین مسیر پویا از گره ابتدایی به گره انتهایی و نتیجتاً مقادیر بهینه تولید در هر پریود استفاده می‌شود.

1. INTRODUCTION

The classical multi-period production-planning problem specifies a discrete-time finite horizon single product inventory management problem subject to deterministic time-varying demand that must be satisfied. This problem has been the subject of extensive research since its introduction in the late 1950s until today. When the production cost and the inventory cost of each period are linear, several authors have presented theorems that can reduce the computational effort required in solving the problem. Wagner and Whitin [1], Zabel [2] give results for the no-backlogging case, while Zangwill [3] analyzes the backlog case. Florian and Robillard [4] have constructed a branch and bound algorithm for solving the concave cost network flow problem with capacity constraints. Many generalizations of the basic model have been considered including introducing bounds on inventory and/or production capacity as well as generalizations to multi-product settings; see Bahl et al. [5] for a review of relevant literature.

In the last decade, two important papers, Aggarwal and Park [6] and Federgruen and Tzur [7] improved the time complexity for obtaining an optimal solution from $O(T^2)$ to $O(T \log T)$, if T represents the length of the planning horizon. Lee et al. [8] studied the multi-period production-planning problem with demand time windows, during which a particular demand can be satisfied with no penalty, and provided polynomial time algorithms for computing its solution.

This paper develops a method for solving the single product multi-period production-planning problem and obtaining the optimal production scheduled for each period, in which the production cost of each period is linear and the inventory cost of each period is concave and backlogging is not permitted. We then extend this assumption to some more general setting, in which the production cost of each period can be concave. It is also assumed that the unit variable cost of the production evolves according to a continuous time Markov process.

When the unit variable cost of the production in each period is a constant value, the production-planning problem can be stated as a problem of finding the shortest path from the source node to the sink node. In this case, the standard shortest path algorithms can help us to solve the problem. When the unit variable cost of the production evolves according to a continuous time Markov process, we should find the dynamic shortest path from the source node to the sink node for solving the problem.

In this paper, we apply the stochastic dynamic programming to find the dynamic shortest path from the source node to the sink node and obtain the optimal production scheduled for each period. This approach for solving the dynamic multi-period production planning problem is a new approach and we could not find the similar papers in this area.

In section 2, we describe the framework of the proposed method. In section 3, we solve a numerical example, and finally we draw the conclusion of the paper.

2. FRAMEWORK OF THE PROPOSED METHOD

Let X_t represent the production scheduled in period t $(t=1,2,..,T)$, D_t represent the expected demand in period t, I_t represent the net inventory at the end of period t (assuming $I_0=I_T=0$), $H_t(I_t)$ represent the inventory cost in period t which is a concave function and c_t represent the unit variable cost in period t which evolves according to a continuous time Markov process under the following assumptions.

1. Assume that the number of states of the unit variable cost is equal to N (these states are in this order: c^1, c^2,.., c^N) and p_{im} represents the probability of transition of this cost from c^i to c^m.
2. Let t_{im} represent the duration of time that the unit variable cost is c^i, before transition to c^m. Its density function is equal to $f_{im}(t)$. It is clear that $w_i(t)$ or the density function of the staying time in state c^i is computed from this equation:

$$w_i(t)=\sum_{m=1}^{N} p_{im} f_{im}(t) \qquad (1)$$

3. Let $\phi_{im}(t)$ represent the conditional probability that the unit variable cost becomes c^m, given that at time zero, it was c^i.

How can a process that started by entering state c^i at time zero be in state c^m at time t. One way this can is for c^i and c^m to be the same state and for the process never to have left state c^i throughout the period $(0,t)$. This requires that the process make its first transition after time t. Every other way to get from state c^i to state c^m in the interval $(0,t)$ requires that the process make at least one transition during that interval. For example, the process could have made its first transition from state c^i to some state c^l at a time τ, $0<\tau<t$, and then by some succession of transitions have made its way to state c^m at time t. These considerations lead us to Equation 2 for computing $\phi_{im}(t)$.

$$\phi_{im}(t)=\delta_{im}\int_t^{\infty} w_i(\tau)d\tau+\sum_{l=1}^{N} P_{il}\int_0^t f_{il}(\tau)\phi_{lm}(t-\tau)d\tau$$

$$\delta_{im}=1 \quad \text{if } i=m$$

$$\delta_{im}=0 \quad \text{otherwise} \qquad (2)$$

Certainly we cannot directly compute $\phi_{im}(t)$ from Equation (2), but since the second integral of Equation (2) is a convolution of the two functions, we can compute $\phi_{im}(t)$ by the Laplace transform. Let $\phi_{im}^e(s)$ represent the Laplace transform of

$\phi_{im}(t)$ that is computed in this manner:

$$\phi^e_{im}(s)=\delta_{im}\int_0^\infty\int_t^\infty e^{-st}w_i(\tau)d\tau\, dt+\sum_{l=1}^{N}P_{il}f^e_{il}(s)\phi^e_{lm}(s) \tag{3}$$

The appropriate model for obtaining the optimal values of X_t is in this manner:

$$MIN\ Z=E\left(\sum_{t=1}^{T}c_tX_t+\sum_{t=1}^{T-1}H_t(I_t)\right)$$

S.T:

$$\begin{aligned}&X_1-D_1=I_1\\&I_{t-1}+X_t-D_t=I_t(t=2,3,...,T-1)\\&I_{T-1}+X_T-D_T=0\\&I_t\geq 0, X_t\geq 0\end{aligned} \tag{4}$$

When the values of c_t (t = 1,2,..,T) are constant, it is proved that an optimal program has the property that production in any period t must be one of these values: 0 or $\sum_{j=t}^{k}D_j$, for k = t, t+1, .., T, because X_tI_{t-1} = 0 (see Johnson and Montgomery [9] for more details). For example, Figure 1 is a network representing such programs for T = 5. Arc (j,k) represents a decision to supply the requirements for period j+1, j+2, .., k by production in period j+1.

If M_{jk} represents the cost of arc (j,k) which includes production and inventory costs, arc (j,k) could be assumed to have length M_{jk} and the production planning problem could be stated as a problem of finding the shortest path from node 0 to node T.

Now, we consider the case that the unit variable cost of the production evolves according to a continuous time Markov process. In this case, the structure of constraints does not change. Then

$$\begin{aligned}&X_{j+1}=D_{j+1}+D_{j+2}+...+D_k\\&I_t=X_{j+1}-\sum_{r=j+1}^{t}D_r=\sum_{r=t+1}^{k}D_r(t=j+1,j+2,...,k-1)\\&I_j=I_k=0\end{aligned} \tag{5}$$

Now, if the unit variable cost of the production at the beginning of period j+1 is c^i, we have these relations:

$$\begin{aligned}&M_{jk}=c^iX_{j+1}+\sum_{t=j+1}^{k-1}H_t(I_t)\\&M_{jk}=c^i\sum_{r=j+1}^{k}D_r+\sum_{t=j+1}^{k-1}H_t(\sum_{r=t+1}^{k}D_r)\end{aligned} \tag{6}$$

Let A(j) represent the set of adjacent nodes of node j. It is clear that A(j)={j+1, j+2, .., T}. Let $V_j(c^i)$ represent the minimum of production and inventory costs from period j+1 to T, if the unit variable cost at the beginning of period j+1 is c^i.

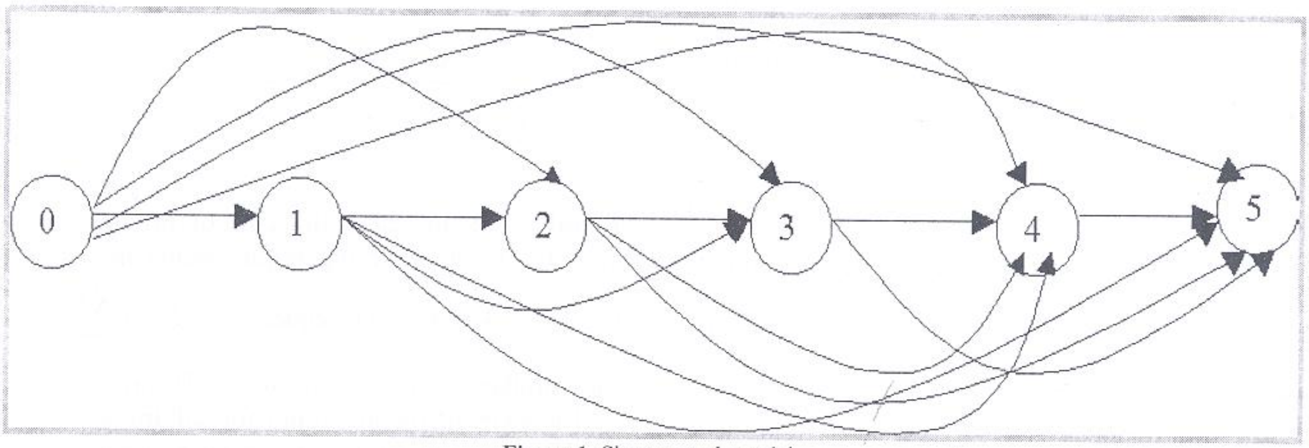

Figure 1. Shortest path model.

Theorem 1. $V_j(c^i)$ for I = 1, 2, .., N can be obtained from the recursive functions (7).

$$V_j(c^i) = \min_{k\in A(j)}\left\{c^i \sum_{r=j+1}^{k} D_r + \sum_{t=j+1}^{k-1} H_t(\sum_{r=t+1}^{k} D_r) + \sum_{m=1}^{N}\phi_{im}(k-j)V_k(c^m)\right\}$$
i = 1,2, ..,N

(7)

Proof. If the unit variable cost of the production at the beginning of period j+1, taking into account the state variable of the system is c^i and we decide to satisfy requirements in periods j+1,j+2, ..,k ($k\in A_j$) by production in period j+1, the costs of production and inventory from period j+1 to k would be

$$c^i \sum_{r=j+1}^{k} D_r + \sum_{t=j+1}^{k-1} H_t(\sum_{r=t+1}^{k} D_r) \tag{8}$$

The probability that after k-j periods, the unit variable cost of the production becomes c^m, given that at the beginning of period j+1 this cost was c^i, would be $\phi_{im}(k-j)$, because the continuous time Markov process corresponding to the transitions of the unit variable cost is memoryless. Now, by conditioning on the state of the unit variable cost after k-j periods, it is proved that the cost of production and inventory from period k+1 to T would be

$$\sum_{m=1}^{N}\phi_{im}(k-j)V_k(c^m) \tag{9}$$

and Theorem 1 is proved. □

The steps of this algorithm are as follows:

Step 1. Begin from node j=T. It is clear that $V_T(c^i)=0$ for all values of i.

Step 2. Set j=j-1. Then compute $\phi_{im}(k-j)$ for all values of i, m and $k\in A_j$ by getting the inverse Laplace of $\phi_{im}^{e}(s)$ in Equation (3).

Step 3. Compute $V_j(c^i)$ for all values of i, from the recursive functions (7). If $V_j(c^i)$ is obtained from $k\in A_j$, the optimal value of production, taking into account c^i would be $X_{j+1}=\sum_{r=j+1}^{k} D_r$.

Step 4. If j>0, then go to 2. Otherwise go to 5.

Step 5. Stop. The optimal values of production in any period t=1,2, ..,T taking into account the state of the unit variable cost of the production at the beginning of period t was obtained.

The time complexity of this algorithm in step 2 is $O((T-1)N^2)$, because the number of states of the unit variable cost is equal to N, and we should repeat this step T-1 times. The time complexity of this algorithm in step 3 is $O\left(T(T+1)N/2\right)$, because in each node j=0,1, ..,T-1, we have T-j combinations for all values of $V_j(c^i)$, for i=1,2, ..,N, and $\sum_{j=0}^{T-1}(T-j) = T(T+1)/2$. Therefore, the time complexity of this algorithm in these steps is polynomial.

We can also extend the assumptions of this paper in order to solve the multi-period production-planning problem with concave production cost. Let $c_tF_t(X_t)$ represent the production cost of period t, in which $F_t(X_t)$ is a concave function and c_t is its cost coefficient. For example, $c_tF_t(X_t)$ can be given by

$$c_tF_t(X_t) = c_t\sqrt{X_t} \tag{10}$$

Now, we assume that the cost coefficient evolves according to a continuous time Markov process. Let $V_j(c^i)$ represent the minimum of production and inventory costs from period j+1 to T, if the cost coefficient at the beginning of period j+1 is c^i.

Corollary 1. $V_j(c^i)$ for i=1,2, ..,N can be obtained from the recursive functions (11).

$$V_j(c^i) = \min_{k\in A(j)}\left\{c^i F_{j+1}(\sum_{r=j+1}^{k} D_r) + \sum_{t=j+1}^{k-1} H_t(\sum_{r=t+1}^{k} D_r) + \sum_{m=1}^{N}\phi_{im}(k-j)V_k(c^m)\right\}$$
i=1,2, ..,N

(11)

Proof. In this case, the cost of production in period j+1 for satisfying requirements in periods j+1,j+2, ..,k ($k\in A_j$) is equal to $c^iF_{j+1}(\sum_{r=j+1}^{k} D_r)$, and Corollary 1 is proved the same as Theorem 1.***

The steps of the algorithm for solving this new problem is the same as the previous model, except that we should refer to the recursive functions (11) instead of the recursive functions (7), and also the time complexity of the related algorithm is polynomial.

3. NUMERICAL EXAMPLE

Production is to be planned for a five-period horizon. There is no initial inventory and the final inventory level is to be zero. The production and inventory costs have the following forms:

$$c_t F_t(X_t) = c_t \sqrt{X_t}$$
$$H_t(I_t) = h_t I_t \tag{12}$$

It is assumed that the cost coefficient evolves according to a continuous time Markov process with two states, $c^1=150$ and $c^2=200$ whose transition matrix is

$$P = \begin{bmatrix} 0 & 1 \\ .4 & .6 \end{bmatrix}$$

It is also assumed that the values of $f_{im}(t)$ are as follows:

$$f_{11}(t) = f_{12}(t) = f_{21}(t) = f_{22}(t) = e^{-t} \quad t > 0 \tag{13}$$

Estimates of h_t and D_t are given in Table 1.

The network corresponding to this numerical example is shown in Figure 1. The results of solving this numerical example have been summarized in Table 2.

Therefore, in each period, we should produce as equal as its demand in both states. It is because of the great values of the unit holding costs per each period t, in this numerical example.

4. CONCLUSION

In this paper, we presented an algorithm based on Semi-Markov decision processes and network flows theory to solve the single product multi-period production planning problem, in which the production and the inventory costs of each period are concave and backlogging is not permitted and also the unit variable cost of the production evolves according to a continuous time Markov process.

It was proved that this production-planning problem could be stated as a problem of finding the dynamic shortest path from the source node to the sink

TABLE 1. Data For the Numerical Example.

t	D_t	h_t
1	100	40
2	200	50
3	150	30
4	180	20
5	200	30

TABLE 2. Results of Solving the Numerical Example.

t	$V_{t-1}(c^1)$	$V_{t-1}(c^2)$	X_t^* (if $c_t=c^1$)	X_t^* (if $c_t=c^2$)
5	2121.3	2828.4	200	200
4	4515.6	5356.1	180	180
3	6806.6	7620.7	150	150
2	9367.5	10270	200	200
1	11354.9	12071.5	100	100

node. Finally, we used the stochastic dynamic programming to find the dynamic shortest path from the source node to the sink node and obtained the optimal production scheduled for each period. Finally, we proved that the time complexity of this algorithm in both cases (linear production cost or concave production cost) is polynomial.

This model can be extended in these directions:

1. Solving the multi-product multi-period production-planning problem.

2. It can be considered that some other parameters of the model, like the unit holding cost per period, evolve according to the semi-Markovian processes.

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