NUMERICAL EVALUATION OF HYDRAULIC FRACTURING PRESSURE IN A TWO-PHASE POROUS MEDIUM

A. Nouri and A. Pak

Department of Civil Engineering, Sharif University of Technology
Tehran, Iran, pak@sina.sharif.ac.ir

(Received: September 30, 1999 – Accepted in Revised Form: May 3, 2001)

Abstract Hydraulic fracturing is a phenomenon in which cracks propagate through the porous medium due to high pore fluid pressure. Hydraulic fracturing appears in different engineering disciplines either as a destructive phenomenon or as a useful technique. Modeling of this phenomenon in isothermal condition requires analysis of soil deformation, crack and pore fluid pressure interactions. In this paper a numerical scheme is presented for analysis of soil stresses and deformations and fluid flow in a coupled manner, which is also capable to detect the initiation of fracture in the medium. Applications of the model are shown by illustrative examples.

Key Words Hydraulic Fracturing, Finite Element Method, Effective Stress Analysis, Mohr-Coulomb Criterion, Hyperbolic Model

چکیده شکست هیدرولیکی (آب شکافت) پدیده‌ای است که در اثر فشار زیاد سیال منفذی موجب بوجود آمدن و توسعه ترک در محیط های متخلخل مانند سنگ و خاک می‌گردد. پدیده شکست هیدرولیکی در شاخه های گوناگون مهندسی می‌تواند بصورت یک عامل مخرب و یا بصورت یک فن‌آوری مفید و سودمند مطرح گردد. مدلسازی عددی این پدیده در شرایط دمای ثابت نیازمند تحلیل اندر کنش تغییر شکل خاک، فشار سیال منفذی و ترک خوردگی می‌باشد. در مقاله حاضر یک مدل عددی جهت تحلیل تنش- تغییر شکل و همچنین جریان سیال منفذی بصورت کاملا همبسته ارایه شده است که قادر است رخداد ترک یا شکاف را نیز در محیط متخلخل ارزیابی نماید. کاربرد های مدل عددی تهیه شده با مثالهای متعدد مورد بحث قرار گرفته است.

1. INTRODUCTION

Hydraulic fracturing refers to creation of crack in porous media due to high pore fluid pressure. Hydraulic fracturing can be a destructive natural phenomenon or a technique used in some engineering disciplines to alter the characteristics of soils. In case of earth dams, impounding of the reservoir behind the dam, increases the amount of pore pressure in dam's core and foundation. Pore water pressure, may become high enough to cause crack in the dam leading to complete failure. On the other hand hydraulic fracturing technique is also used, as a tool, to change the flow characteristics of the ground. For example for increasing the extraction of oil from oil wells excavated in heavy oil deposits, steam at high temperature and pressure is injected into the well-bores in order to cause crack in the formation, resulting lower oil viscosity and higher soil permeability which enhance the productivity of the oil well. Other applications of hydraulic fracturing are in situ stress determination (in rock engineering), grouting, soil vapor extraction, etc.

Analysis of hydraulic fracturing has been on empirical basis for a long time. Although experimental studies for understanding the mechanism of this phenomenon (time and place of fracture initiation, and the pattern of fracture propagation) have been conducted by several researchers, modeling of this phenomenon has not been well developed yet. True modeling of hydraulic fracturing requires accounting for the interaction between pore fluid flow and solid deformation. This necessitates that two basic conservation laws for fluid flow and applied loads be solved simultaneously. In this paper, these two equations and their finite element forms will be presented briefly, then application of a code named CFEAP, which is developed specifically for hydraulic fracture modeling will be illustrated.

2. EARLIER WORKS

There is a large body of literature on the topic of fluid flow in porous media without incorporating the effects of induced fractures. Terzaghi [1] was the first who proposed a mathematical theory for a one dimensional consolidation problem. Biot [2,3,4] generalized Terzaghi's theory and analyzed a problem of three-dimensional consolidation in anisotropic porous media. Sandhu [5] developed the first finite element formulation for the two-dimensional consolidation problem. Christian [6] presented a finite element solution for stress analysis in soil layer in undrained condition. Later he and Boehmer [7] extended these ideas and developed the finite element formulation for consolidation analysis. Small, Booker and Davis [8] used the principle of virtual work to formulate the finite element consolidation equations of a saturated soil with elastoplastic stress-strain behavior. This work was extended by Carter et al. [8,9] to include finite deformations. Lewis et al. [10] assumed a hyperbolic stress-strain model for the soil and used a nonlinear law for soil permeability in their formulation for modeling the consolidation.
There are few contributions on the numerical modeling of hydraulic fracturing in the geo technical literature. Modeling of hydraulic fracture was first attempted in the field of petroleum reservoir engineering. First generation models of hydraulic fracturing were pioneered by Zheltov and Khristianovich [11], Perkins and Kern [12], and Geertsma and deKlerk [13]. They provided closed form solutions for predicting fracture length and width based on a prescribed geometry for a planar fracture.

Settari and Raisbeck [14,15] developed two of the early models for simulating hydraulic fracture during cyclic steam stimulation in oil sands. Their model was a two-dimensional finite difference model for single-phase compressible fluid flow in a linear elastic porous material. Settari et al. [16] investigated the effects of soil deformations and fracture on the reservoir in a partially coupled manner. Advani et al. [17] developed a finite element program for modeling three-dimensional hydraulic fractures in multi-layered reservoirs. In this work, propagation of a tensile planar hydraulic fracture in layered reservoirs with elastic behavior was investigated. Pak [18] developed a thermal hydro mechanical finite element model, which was able to detect the hydraulic fracture initiation and its general pattern of propagation.

3. BASIC FORMULATION FOR A COUPLED HYDRO-MECHANICAL ANALYSIS

There are basically two field equations, which must be satisfied, in a coupled hydro-mechanical analysis. These equations are equilibrium equation and continuity equation. The equilibrium equation for a continuum can be written as:

$$\delta\sigma_{ij,j} + \delta F_i = 0 \tag{1}$$

In numerical implementation, the equation is expressed in incremental integral form. Using the weighted residual method and substituting total stress with effective stress and pore fluid pressure yields:

$$-\int_v \delta\sigma'_{ij} w_j dV + \int_v m\delta P w_j dV + \int_s \delta t_{si} w ds + \int_v \delta F_i w dV = 0 \tag{2}$$

By discretization of the domain and taking nodal displacements and pore pressures as state variables, equation (2) leads to the following form:

$$- \int_v [B]^T [D] [B]\, dV\, [\delta U^*] + \int_v [B]^T [m] [N_P] [\delta P^*]\, dV + \int_s [N]^T [\delta T_S]\, dS + \int_v [N]^T [\delta F]\, dV = 0 \tag{3}$$

When considering fluid flow in a porous medium, the continuity equation is given by [19]:

$$\nabla.(\rho v) - G\rho = -\frac{1}{V_t}(V_t\,\rho)^{\bullet} \tag{4}$$

In this equation the compressibility of solid grains is considered to be small and is not taken into account. Equation (4) can be expressed in integral form as:

$$-\int_v v_i w_{,i} dV + \int_s \overline{v_i} n_i w ds + \int_v \overset{\bullet}{\varepsilon}_v w dV - \int_V G w dV = 0 \tag{5}$$

in Equation 5 it is assumed that fluid is incompressible and flow is laminar.

By discretization of the domain and substituting quantities in terms of state variables, the equation (5) leads to the following form [19]:

$$\int_v [B_p]^T K_{i3} dV + \int_v [B_p]^T K_{ij} [B_p] \frac{dV}{\gamma} [\overset{*}{P_t}] +$$
$$\int_s [N_p]^T \bar{v}_i n_i ds + \int_v [B_p]^T K_{ij} [B_p] \frac{dV}{\gamma} \theta [\delta \overset{*}{P}] +$$
$$\int_v [N_p]^T [m]^T [B] \frac{dV}{\Delta t} [\delta \overset{*}{U}] - \int_v [N_p]^T G dV = 0 \qquad (6)$$

Where θ is a number between 0 (fully explicit scheme) to 1 (fully implicit scheme). Equations (3) and (6) must be solved simultaneously in order to consider the interaction between two phenomena. The matrix equation for coupling two processes takes the following form [20]:

$$\begin{bmatrix} [K] & -[£] \\ [£]^{\mathrm{T}} & \theta \Delta\, \mathrm{t}[\mathrm{H}] \end{bmatrix} \begin{bmatrix} [\delta \overset{*}{U}] \\ [\delta \overset{*}{P}] \end{bmatrix} = \begin{bmatrix} [\delta F] \\ [R] \Delta\, t \end{bmatrix} \qquad (7)$$

where

$$[K] = \int_v [B]^T [D] [B] dV$$

$$\delta F = \int_s [N]^T [\delta t_s] ds + \int_v [N]^T [\delta F] dV$$

$$[£]^T = \int_v [N_p]^T [m]^T [B] dV$$

$$[H] = \int_v [B_p]^T K_{ij} [B_p] \frac{dV}{\gamma}$$

$$[R] = -\int_v [B_p]^T K_{i3} dv - \int_v [B_p]^T K_{ij} [B_p] \frac{dv}{\gamma} [\overset{*}{P_t}] -$$
$$\int_s [N_p]^T \bar{v}_i n_i ds + \int_v [N_p]^T G dv$$

Solution procedures for equation (7) are described in [21] and [10].

A finite element program called CFEAP has been developed in FORTRAN using the above formulation. Numerical results obtained from CFEAP have been tested against analytical solutions of one and two dimensional consolidation problems and the results proved to be satisfactory. Validation of this model has completely been documented elsewhere [19].

4. HYDRAULIC FRACTURE MODELING

In numerical analysis, different approaches are used for simulation of fracture, namely, smeared, dual porosity and discrete approach. In smeared approach modified material characteristics (e.g. modified stiffness, permeability, etc.) in the fractured zone are used in the analysis without introducing real fracture. This approach is used when fracturing is so intense that a uniformly damaged material called fractured zone can represent the whole medium or certain part of it. Therefore the basic assumptions of continuum mechanics hold for smeared approach. On the other hand, discrete approach is used where few fractures exist. In this approach after fracturing, a discontinuity is introduced in the medium and the assumption of continuum is no longer valid. Dual porosity approach is basically used for simulation of flow in naturally fractured reservoirs. This type of reservoir in theory is modeled as blocks of solid materials stacked over each other with low porosity and low permeability with a network of fractures between the blocks with high permeability, which dominates the flow regime [18].

There are three main categories of fracture initiation criteria. The first class is the fracture mechanics criteria, second is the bifurcation criterion and those originated from theory of plasticity, and the third class is empirical criteria based on the laboratory experiments. Fracture mechanics theories, which were originally developed for metals have been used successfully for geological materials in recent years [22]. Linear elastic fracture mechanics (L.E.F.M.) or elastoplastic fracture mechanics (E.P.F.M) have been used for analyzing fractures in soils and rocks [14-29-30]. Modeling of fracture requires knowledge of geological conditions in the ground. Local stress field and variation of stresses between adjacent formations are often thought to be the main factors, which control fracture orientation and fracture growth. Regional stresses in the ground can have an impact on the azimuthal trend of the hydraulically induced fractures. It is usually believed that the fracture propagates perpendicular to the direction of the minimum principal stress, i.e., tensile fracture is the prime mechanism in hydraulic fracturing. Recently the possibility of shear failure before tensile failure has been investigated especially where the fluid leak-off into the formation is significant. Other class of criteria for crack initiation and crack growth is the fracture criteria based on the theory of plasticity. Because of ambiguities in the hydro-fracturing phenomenon in geomaterials, validity of usage of the plasticity theories to obtain suitable fracture criteria should be carefully examined. Finally criteria based on laboratory studies can be used for fracture initiation and growth, however, this type of criteria are valid for special kind of soil under consideration.

Importance of performing a deformation analysis for fracture modeling emerges here. Fracturing criteria are based on the stresses and deformation in the ground. Therefore, in order to obtain a realistic model for design purposes, the behavior of the ground has to be accounted for in any realistic hydraulic fracturing analysis.

In this study the occurrence of hydraulic fracture has been detected by the onset of shear/tensile crack in the medium based on the latest stress state calculated in the medium. No attempt has been made to model the propagation of fracturing therefore no fracture element has been inserted in the finite element mesh.

A finite element code has been developed in FORTRAN, which solves two basic equations of equilibrium and fluid continuity in a fully coupled manner. What distinguishes this software from its counterparts is its capability to detect the initiation of fracture in the medium. CFEAP solves problems in plane strain condition. Two types of elements have been coded in the program: 4-node and 8-node quadrilateral elements. Constitutive laws that are coded in the program are linear elastic (in which the Young's modulus is increased with increasing the minimum effective stress, in order to simulate the natural ground conditions where the modulus usually increases with depth) and nonlinear hyperbolic model proposed by Duncan and Chang [20]. CFEAP has the capability of applying different fracture criteria to detect the initiation of fracture. A user-defined fracture criterion can be inserted in the code in order to determine the initiation of fracture based on the latest calculated stress state in the medium. Since fracture mode is not clear in advance, two criteria for tensile and shear fracture has been coded in the program as defaults. For tensile fracture the criterion can simply be a zero or negative (tensile) stress in the medium. Whenever the minimum effective stress at a node becomes zero or negative, the program can detect the node in which the initiation of fracture is likely. If the material is cohesive, the amount of cohesion may be inserted in the model as a flag to detect the nodes in which the tensile stresses are less than the material cohesion. For shear fracture, the criterion is a Mohr-Coulomb type relation. At the end of each time step, when the program calculates the stress components, it is checked, at any node, whether or not this relation is satisfied. In this way the nodes prone to shear fracture are recognized by the program. It should be noted that the criterion that is satisfied first is considered to be the cause of fracture.

5. MODELING OF ISOTHERMAL HYDRAULIC FRACTURING EXPERIMENTS

In this part validation of the model and its capabilities to simulate the experimental studies of hydraulic fracturing is examined. Komak Panah and Yanagisawa [23] for obtaining a criterion for hydraulic fracturing pressure arranged some experiments on hollow cylindrical sand specimens with different all around pressures, and applied inside pressures until fracturing occurred. The cross section of the specimen is shown in Figure 1. This is a hollow cylindrical specimen with internal radius of 2cm and external radius of 5cm and height of 12cm. Outside surface of the specimen is free draining under atmospheric pressure conditions. In these experiments, fracturing was identified based on the amount of the water passed through the sample. An immediate increase of the water passed through the specimen indicated the occurrence of hydraulic fracturing and final destruction of the specimen. The experiments had been done with confining pressures of 0.5, 1, 2, and 3 kg/cm^2. The same confining pressures have been used in the numerical analysis.

Material parameters, which were compacted

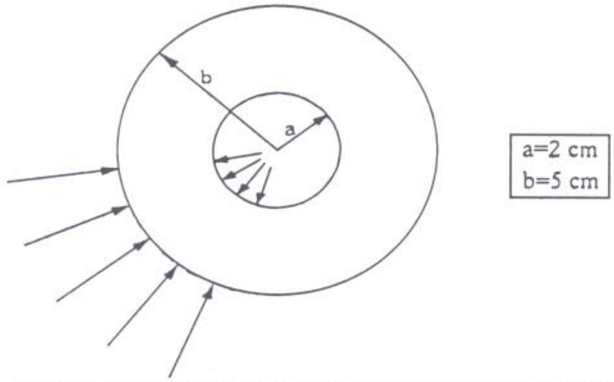

Figure 1. Sample section used in experiments performed by Komak-panah and Yanagisawa.

silty clay are as follows [23]:

LL = 49.28%
PI = 24.28%
O.M.C. = 36%
$\phi_u = 6.5°$

PL = 25.54%
$G_S = 2.71$
$C_U = 0.11\ \mathrm{kg}/_{\mathrm{cm}}{}^{2}$
$C' = 0.05\ \mathrm{kg}/_{\mathrm{cm}}{}^{2}$

6. FINITE ELEMENT MODELING

The conducted hydraulic fracturing experiments have been simulated using the program CFEAP. Section of the specimen used in the hydraulic fracturing test was discretized as shown in Figure 2.

Hydraulic fracturing pressure in two states, namely initiation of fracture and final destruction of the specimen was obtained. Hydraulic fracturing pressure was obtained by trial and error, which required a great number of analyses. It was assumed that fracturing initiates when only one node in the finite element mesh satisfies the maximum tensile stress criterion for tensile fractures or a Mohr-Coulomb type criterion for shear fracture. Final failure occurs when one of the above-mentioned criteria is satisfied in all Gauss points (integration points) throughout the thickness of the specimen.

Time marching is carried out for solving the equations and at the end of each time step pore pressures and displacements are updated. It should be noted that in any coupled analysis of consolidation, the choice of time increment is crucial and unsuitable Δt may render instability in some cases. Time increment should not be large due to the limitations of equilibrium equation and on the other hand, it should not be too small because of continuity equation. For choosing the appropriate Δt for consolidation problems some criteria has been proposed [24,25]. For improving the results in some cases a fully implicit scheme ($\theta = 1$) has been adopted.

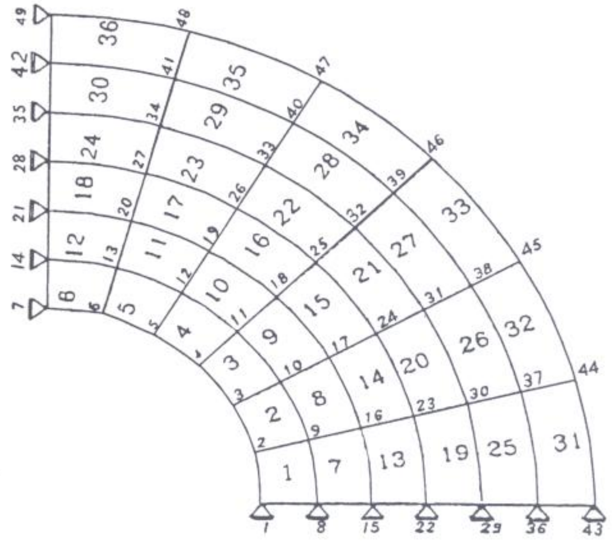

Figure 2. Finite element mesh of sample section.

7. RESULTS OF THE NUMERICAL SIMULATION OF LABORATORY TESTS USING ELASTIC MODEL

By using CFEAP, a number of numerical simulations with elastic model have been performed. Modulus of elasticity in the elastic model is variable with minimum effective stress and the parameters of the model are the same as those of hyperbolic model, which will be presented later.

Numerical simulation have been performed for 4 different states of all around pressures i.e. 0.5, 1.0, 2.0, and 3.0 kg/cm2 (approximately equal to 50-300 N/cm2), and hydraulic fracturing pressures have obtained for initiation of fracture as well as final destruction of the sample. Figure 2 shows the finite element mesh assuming plane strain condition.

Figure 3 shows a comparison between experimental results and numerically obtained pressure for initiation of hydraulic fracturing. High conformity between these two results is interesting. Pi in Figure 3

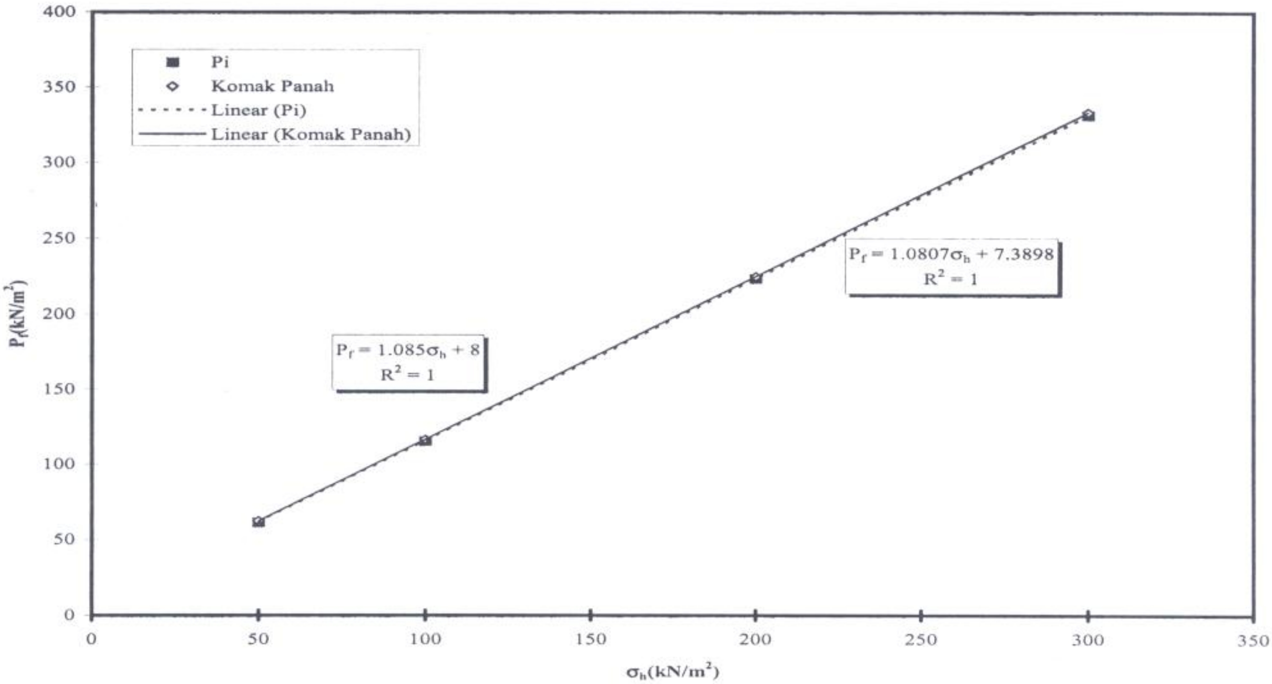

Figure 3. Comparison between experimental results and numerically obtained pressure for initiation of hydraulic fracture using elastic model.

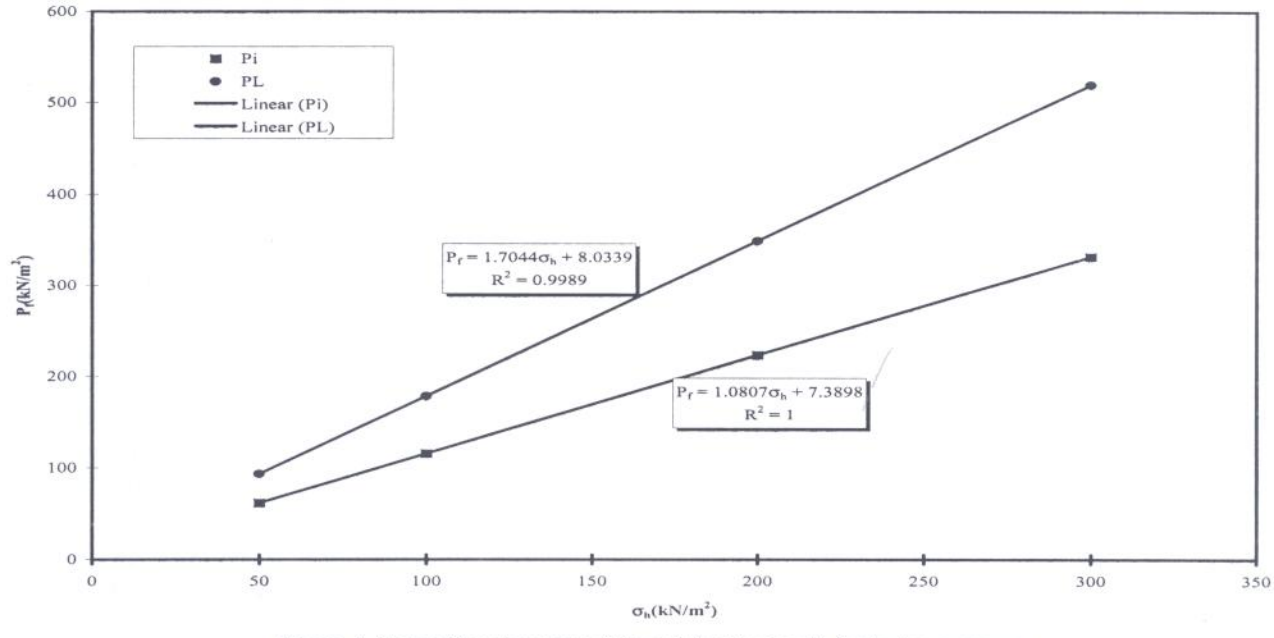

Figure 4. Comparison between initial and final hydraulic fracturing pressures obtained by CFEAP using elastic model.

represents the pressure required for initiation of hydraulic fracture based on numerical modeling.

Figure 4 shows a comparison between initial and final hydraulic fracturing pressures, both of which are obtained from numerical simulation. It is clear that although the results of final hydraulic fracturing pressure (specimen destruction) with different confining pressures also make a straight line, but the difference between initial and final pressures (for crack initiation and a crack throughout the thickness of the specimen) is appreciable. Pi and Pl are pressures required for initiation of hydraulic fracture and final destruction of the specimens respectively.

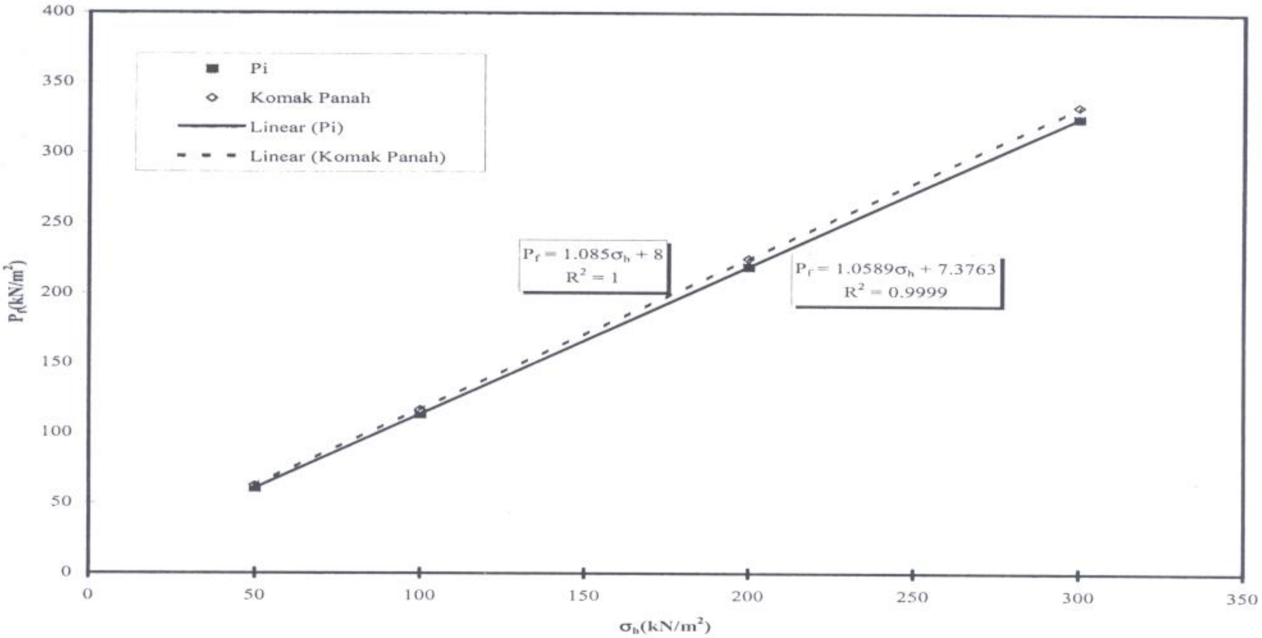

Figure 5. Comparison between experimental results and hydraulic fracture initiation pressure obtained by CFEAP using hyperbolic model.

8. RESULTS OF THE NUMERICAL SIMULATION OF LABORATORY TESTS USING HYPERBOLIC MODEL

A number of simulations have also been performed using hyperbolic model to determine initial and final pressure of hydraulic fracturing. Model parameters are as follows: $k = 600$; $n = 0.45$; $R_f = 0.7$; $k_b = 50$; $m = 0.2$; $\Delta\Phi = 0$ (These parameters are defined at the end of the paper).

Finite element mesh is the same as that used in the elastic solution (Figure 2). Numerical simulation has been performed in four different conditions of confining pressure of 50, 100, 200 and 300 N/Cm^2 and initial and final pressure of hydraulic fracturing have been obtained. Figure 5, shows a comparison between experimental results and the pressure corresponding to the initiation of hydraulic fracturing obtained by numerical modeling. Similarity between these two results is remarkable.

Figure 6 depicts a comparison between initial and final pressure of hydraulic fracturing both obtained from numerical modeling. It is very important to note that contrary to the elastic model, difference between initial and final pressure of hydraulic fracturing in this case is negligible. This is more consistent with the observations during the experimental studies.

Figure 7 shows a comparison between the experimental results and final pressure of hydraulic fracturing obtained from numerical solution. In this case final pressure of hydraulic fracturing obtained using hyperbolic model is conformable with laboratory results.

Figure 8 shows the pressures required for initial and final hydraulic fracturing both obtained from numerical modeling using hyperbolic model along with the hydraulic fracture pressures recorded during the experiments. Similarity among all results is remarkable.

9. CONCLUSIONS

According to the importance of hydraulic fracturing phenomenon in civil and reservoir engineering fields, and noting that there is no commercial software available that is able to simulate the hydraulic fracturing process (initiation and propagation), a numerical model is developed and introduced in this paper. The fully coupled hydro-mechanical scheme presented in this paper is capable of analyzing hydraulic fracturing phenomenon in porous media. The proposed scheme can use linear elastic and hyperbolic constitutive models in the analysis. By simulating the hydraulic fracturing

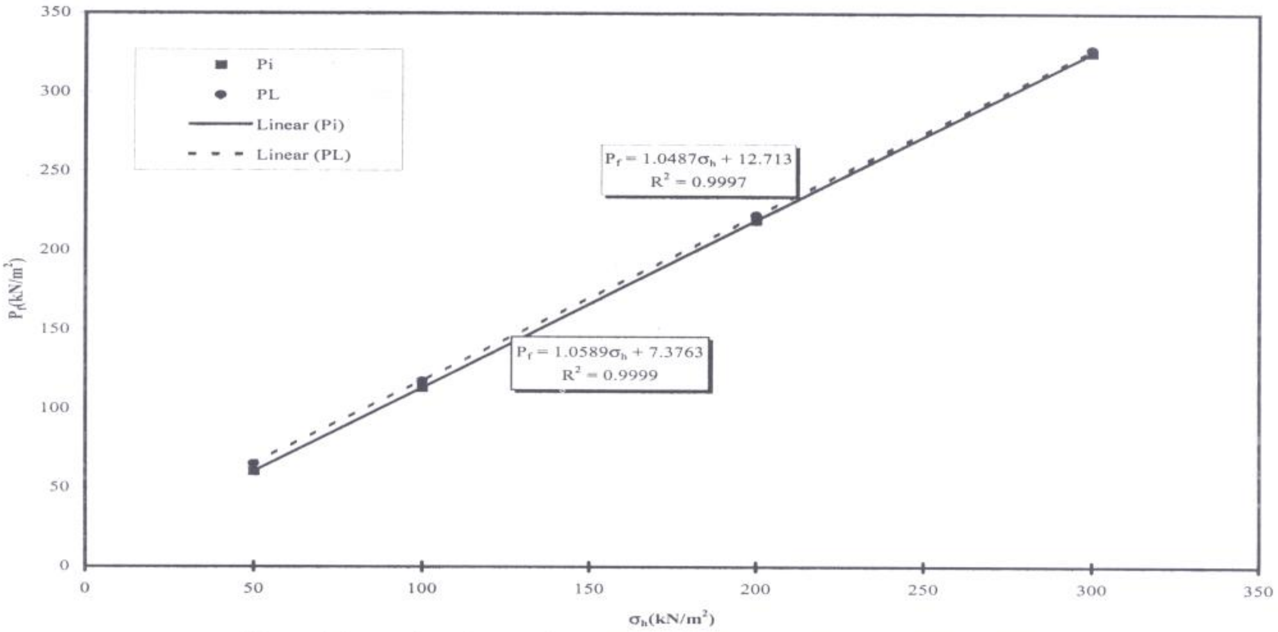

Figure 6. Comparison between initial and final hydraulic fracture pressure obtained by CFEAP using hyperbolic model.

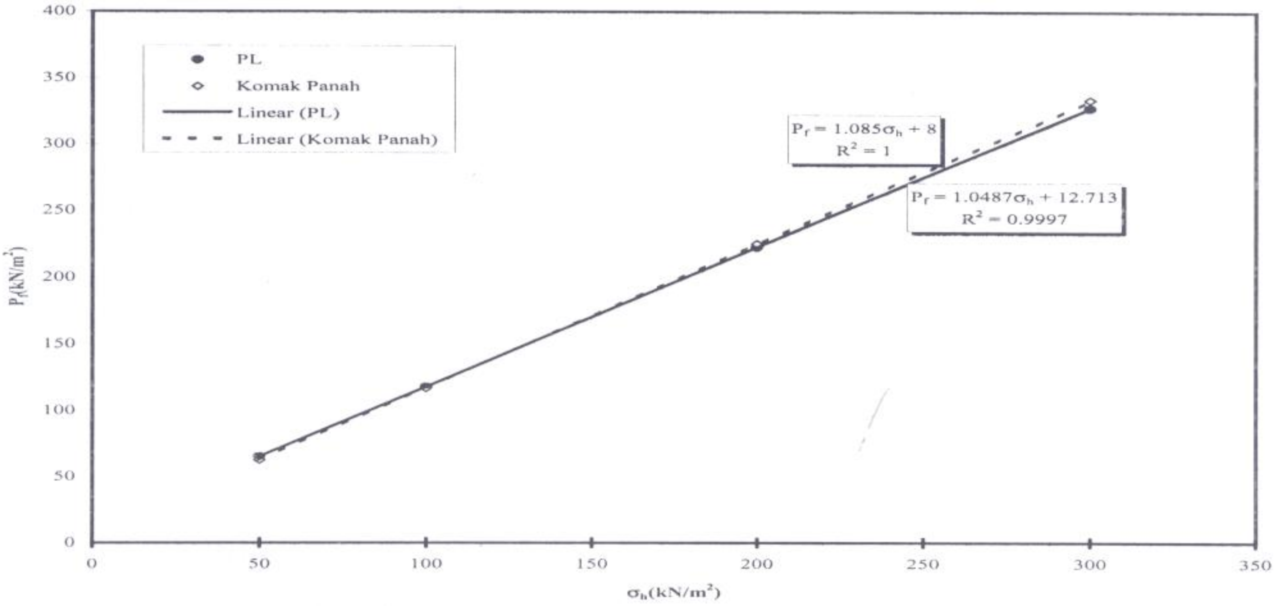

Figure 7. Comparison between experimental results and final pressure of hydraulic fracture obtained by CFEAP using hyperbolic model.

experiments it has been shown that pressures corresponding to the initiation and end of hydraulic fracturing in the elastic analysis are very different, but using hyperbolic model greatly improves the results and decreases the difference considerably. Experimental results demonstrate a very good conformity with the final pressure of hydraulic fracturing using hyperbolic model.

Capability of CFEAP for evaluating the real hydraulic fracturing pressure was demonstrated in this paper. This software can determine the mode of the fracture (tensile or shear) as well as the locations where the possibility of fracture is high. The obtained results confirm that CFEAP has captured the basic aspects of soil fracture behavior. The developed model can be effectively used to find the in-situ stresses in soils/rocks. It can also be used in analyzing earth dams, grouting projects and

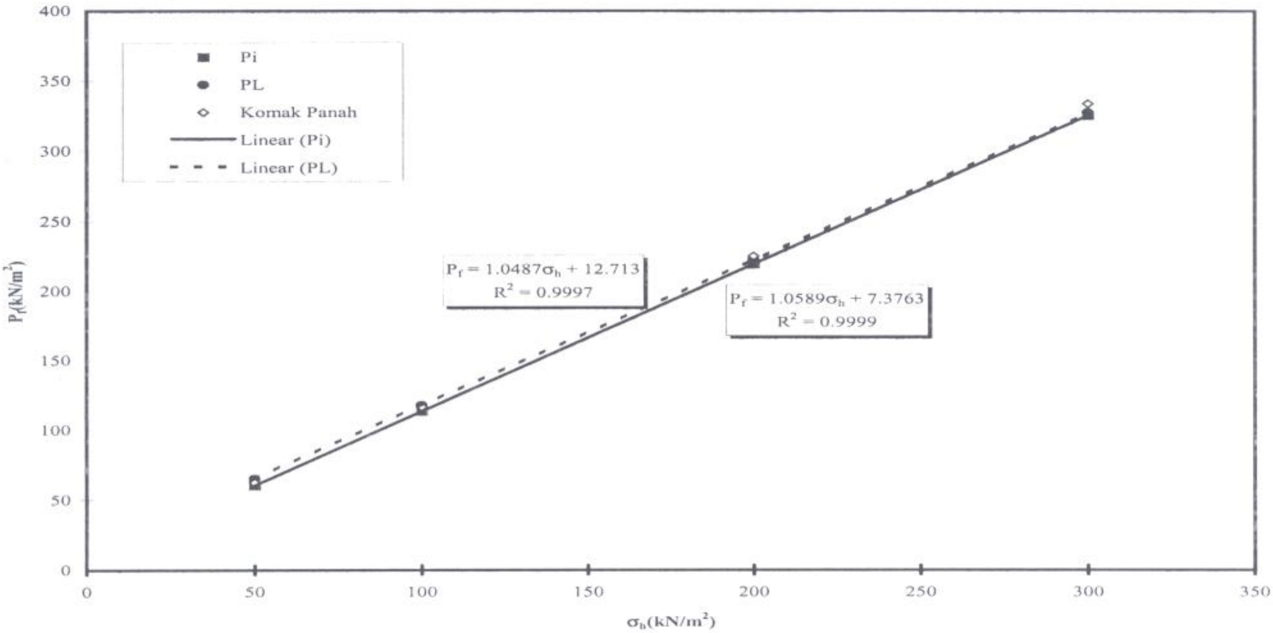

Figure 8. Comparison between experimental results and hydraulic fracturing initiation and final pressures obtained by CFEAP using hyperbolic model.

soil vapor extraction technique to obtain the fluid pressure that causes fracturing in the ground.

10. APPENDIX

Shape functions that have been used in the code for coupled analysis are as follows:

1. For displacements 8-node rectangular elements:

$$[N]=\begin{bmatrix}\Phi_1 & 0 & \Phi_2 & 0 & \Phi_3 & 0 & \Phi_4 & 0 & \Phi_5 & 0 & \Phi_6 & 0 & \Phi_7 & 0 & \Phi_8 & 0\\ 0 & \Phi_1 & 0 & \Phi_2 & 0 & \Phi_3 & 0 & \Phi_4 & 0 & \Phi_5 & 0 & \Phi_6 & 0 & \Phi_7 & 0 & \Phi_8\end{bmatrix}$$

2. For corner nodes:

$$\Phi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1) \qquad i= 1, 2, 3, 4$$

3. For mid-nodes ξ=0:

$$\Phi_i = \frac{1}{2}(1-\xi^2)(1+\eta\eta_i) \qquad i= 5, 7$$

4. For mid-nodes η=0:

$$\Phi_i = \frac{1}{2}(1-\eta^2)(1+\xi\xi_i) \qquad i = 6, 8$$

5. For pore pressures, 4-node or 8-node rectangular elements can be used. For 4-node element:

$$< N_P >=< \Phi_1 \quad \Phi_2 \quad \Phi_3 \quad \Phi_4 \quad 0 \quad 0 \quad 0 \quad 0 >$$

where

$$\Phi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \qquad i =1, 2, 3, 4$$

11. NOTATION

σ_{ij}	Stress tensor at any point
F_i	external loads
t_S	surface traction
i, j	indices taking 1,2 and 3 representing coordinate axis
w	weighting function
m	vector equal to $m = [1 \ 1 \ 1 \ 0 \ 0 \ 0]^T$ in 3D and $m = [1 \ 1 \ 0]^T$ in 2D condition
P	pore fluid pressure
ρ	density of fluid
v	velocity vector of flowing fluid
$\overline{V}_i$	specified velocity vector
G	fluid volume from sink or source
δ	variable increment
∇	gradient operator
V_t	volume
n_i	unit vector normal to the surface of the boundary
K_{ij}	permeability tensor
H	total head
ε_v	volumetric strain
p	pore pressure
N	shape function matrix
B	derivatives of shape functions matrix
N_P	shape function matrix for pore pressure
B_P	derivative of shape functions for pore pressure
D	constitutive matrix
γ	unit weight
U^*	nodal displacements
P^*	nodal pore pressures
C_U	undrained shear strength
ϕ_U	undrained friction angle
G_S	specific gravity
OMC	optimum moisture content

C'	effective cohesion
ϕ'	effective friction angle
k	modulus number (hyperbolic model)
n	modulus exponent (hyperbolic model)
k_b	bulk modulus number (hyperbolic model)
m	bulk modulus exponent (hyperbolic model)
R_f	failure ratio
θ	a value between 0 and 1

12. REFERENCES

1. Terzaghi, K., "Erbaumechanik Auf Bodenphysikalischer Grundlage (Leipzig F. Deuticke 1925)", Principles of Soil Mechanics, Eng. News Record, A Series of Articles, (1925).
2. Biot, M.A., "General Theory of Three Dimensional Consolidation", ***J. Appl. Physics***, Vol. 12, (1941), 155-164.
3. Biot, M.A., "General Solutions of The Equations of Elasticity and Consolidation For A Porous Material", ***J. Applied Mech.***, Trans. ASME, Vol. 78, (1956a), 91-96.
4. Biot, M.A., "Theory of Deformation of A Porous Viscoelastic Anisotropic Solid", ***J. Applied Physics***, Vol. 27, (1956b), 459-467.
5. Sandhu, R.S., "Fluid Flow in Saturated Porous Elastic Media", ***Ph.D. Thesis***, Department of Civil Engineering, University of California, Berkeley, (1968).
6. Christian, J.T., "Undrained Stress Distribution By Numerical Methods", ***J. Of Soil Mech. Found. Eng. Div.***, ASCE, Vol. 94, SM 6, (1968), 1333-1345.
7. Christian, J.T. and Boehmer, J.W., "Plane Strain Consolidation By Finite Elements", ***J. Of Soil Mech. Found. Eng. Div.***, ASCE, Vol. 96, SM 4, (1970), 1435-1457.
8. Carter, J.P., Small, J.C. and Booker, J.R., "A Theory of Finite Elastic Consolidation", ***Int. J. Of Solids Structures***, Vol. 13, (1977), 467-478.
9. Carter, J.P., Booker, J.R. and Small, J.C., "The Analysis of Finite Elastoplastic Consolidation", ***Int. J. Of Num. Anal. Methods in Geomech.***, Vol. 3, (1979), 107-129.
10. Lewis, R.W., Roberts, G.K. and Zienkiewicz, O.C., "A Nonlinear Flow and Deformation Analysis of Consolidation Problems", ***2nd Int. Con. on Num.Methods in Geomechanics***, Blacksburg, Virginia, Vol. 2, (1976), 1106-1118.
11. Zheltov, Y.P. and Khristianovitch, S.A., "On The Mechanism of Hydraulic Fracturing of An Oil-Bearing Stratum", ***Izvest. Akad. Nauk SSR***, OTN, Vol. 5, (1955), 3-41 (in Russian).
12. Perkins, T.K. and Kern, L.R., "Width of Hydraulic Fractures", ***JPT***, Trans., AIME, 222, (Sept. 1961), 937-949.
13. Geertsma, J. And deklerk, F.A., "Rapid Method of Predicting Width and Extent of Hydraulically Induced Fractures", ***JPT,*** Trans., AIME, 246, (Dec. 1969), 1571-1581.
14. Settari, A. And Raisbeck, J.M., "Fracture Mechanics Analysis in In-Situ Oilsand Recovery", ***JCPT***, (April-June 1979), 85-94.
15. Settari, A. And Raisbeck, J.M.,"Analysis and Numerical Modeling of Hydraulic Fracturing During Cyclic Steam Stimulation in Oilsands", ***JPT***, (Nov. 1981), 2201-2212.
16. Settari, A., Kry, P.R. and Yee, C.T., "Coupling of Fluid Flow and Soil Behavior To Model Injection into Uncemented Oilsands"; ***JCPT***, 28(1), 81-92, Jan.-Feb. (1989).
17. Advani, S.H., Lee, T.S. and Lee, J.K., "Three-Dimensional Modeling of Hydraulic Fractures In Layered Media, Part I and Part II", ***Journal of Energy Resources Technology***, Vol. 112, (1990), 1-19.
18. Pak. A., "Numerical Modeling of Hydraulic Fracturing", Ph.D. Thesis, Department of Civil and Environmental Engineering, University of Alberta, Canada, (1997).
19. Nouri, A., "Numerical Study of Hydraulic Fracturing in Two- Phases Porous Media", M.Sc. Thesis, Department of Civil Engineering, Sharif University of Technology, Tehran, Iran, (1998).
20. Duncan, J.M. and Chang, C.Y., "Nonlinear analysis of Stress and Strain in Soils", ***Journal of Soil Mechanics and Foundation Engineering Division***, ASCE, Vol. 96, SM 5, (1970), 1629-1653.
21. Britto, A.M. and Gunn, M.J., "Critical State Soil Mechanics Via Finite Elements", Ellis Horwood Limited, (1987).
22. Atkinson, B.K. (editor), "Fracture Mechanics of Rocks", ***Academic Press***, Geology Series, (1987).
23. Komak- Panah, A. and Yanagisawa, E., "Laboratory study of Hydraulic Fracturing Criteria in Soils", ***Soils and Foundation***, vol. 29. No. 4, (1989), 14-22.
24. Vermeer, P.A. and Verruijit, A., "An Accuracy Condition for Consolidation Analysis by FEM", ***Int. J. for Num. and Anal. Methods in Geomech.*** Vol. 5, (1981), 1-14.
25. Gatmiri, B., "Validation Du Code UDAM, UDAM: A Finite Element Software for the Analysis of Unsaturated Porous Media", ***Internal Report CERMES-ENPC***, Paris, (1993).
26. Christian, J.T., "Two- and Three Dimensional Consolidation", in Numerical Methods in Geomechanical Engineering (eds. C.S. Desai and J.T. Christian), London McGraw-Hill, (1977), 399-426.
27. Settari, A., Raisbeck, J.M., "Fracture Mechanics Analysis in In- Situ Oilsand Recovery", ***JCPT***, April- June, (1979) 85-94.
28. Pak, A., SadrNejad, A., and Nouri, A., "Influence of the Initial Effective Principal Stresses on Hydraulic Fracture in Soils ", ***Scientia Iranica***, Vol. 5, No. 3,4, (1999).
29. Settari, A., Raisbeck, J.M., "Analysis and Numerical Modeling of Hydraulic Fracturing during cyclic steam simulation in oilsands", ***JPT***, Nov. (1981), 2201-2212.
30. Settari, A., "Simulation of Hydraulic Fracturing Process", ***SPE Journal***, (Dec. 1980), 2187-500.
31. Chan, D. and Pak, A., "Numerical Modeling of Hydraulic Fracturing in a Porous medium", ***International Conference on Geomechanics/Ground Control in Mining and Underground Construction***, Wollongong, Australia, (1998).
32. Lewis, R. W., Roberts,G.K. and Zienkiewicz, O.C., "A Nonlinear Flow and Deformation Analysis of Consolidation Problems", ***2nd. Int. Conf. On Num. Methods in Geomech.***, Blacksburg, Virginia, Vol. 2, (1976), 1106-1118.