

# FORWARD POSITION KINEMATICS OF A PARALLEL MANIPULATOR WITH NEW ARCHITECTURE

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**Abstract** The forward position kinematics (FPK) of a parallel manipulator with new architecture supposed to be used as a moving mechanism in a flight simulator project is discussed in this paper. The closed form solution for the FPK problem of the manipulator is first determined. It has, then, been shown that there are at most 24 solutions for FPK problem. This result has been verified by using other techniques such as geometric approach and a numerical method known as polynomial continuation. Numerical examples are performed to display all possible solutions available for the devised manipulator.

**Key Words** Parallel Manipulator, Robot Kinematics, Closed Form Solution, Polynomial Continuation

**چکیده** سینماتیک مستقیم موقعیت یک ربات موازی با طراحی جدید که قرار است به عنوان مکانیزم متحرک یک سیمولاتور پرواز مورد استفاده قرار گیرد در این مقاله بررسی شده است. ابتدا حل بسته برای سینماتیک مستقیم موقعیت این ربات استخراج گردیده و نشان داده شده که حداکثر بیست و چهار جواب برای حل مساله سینماتیک مستقیم موقعیت این ربات وجود دارد. سپس صحت نتایج با روشهای دیگر نظیر روش هندسی و روش عددی معروف به ادامه چند جمله ای نیز مورد تایید قرار گرفته است. در انتها با اجرای مثالهای عددی تمام جوابهای ممکن برای حل سینماتیک مستقیم موقعیت این ربات نشان داده شده است.

## INTRODUCTION

Serial manipulators usually have some merits such as large workspace, dexterity for maneuvering in small spaces, ability for accessing to points at large distances and design simplicity. However, lack of rigidity because of cantilever structure, small payload capacity, operating inaccuracy and poor dynamical performance at high-speed operations are some drawbacks of serial manipulators. To overcome these drawbacks, an alternate type of manipulator, comprising kinematics loops, known as parallel manipulator has been proposed. The main advantages of parallel manipulators, as compared with their counterparts, are greater rigidity causing higher natural frequencies, homogeneous distribution of inertia, higher accuracy and higher load-carrying capacity [1]. The most important types of parallel manipulators are those with moving and fixed platforms connected by means of parallel legs [1-4]. Figure 1

shows two examples of these types of parallel manipulators.

One of the main problems in the kinematics of parallel manipulators is analysis of forward position kinematics (FPK). The FPK solution comprises a system of nonlinear equations with many variables, which is a time consuming problem. The closed form solution for the problem and the upper limit of the number of solutions are

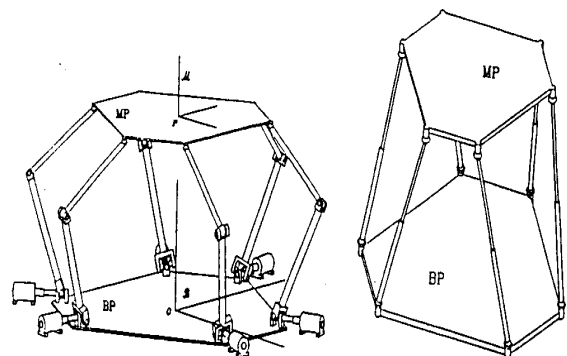


Figure 1. Two examples of parallel manipulators.

the main important issues in the analysis of FPK. The closed form solution is usually a linear polynomial equation in terms of one unknown and can be defined as given input values for the devised manipulator, i.e., the leg lengths and its geometry, determine all possible configurations of the moving platform. There are other issues in kinematics analysis besides those mentioned investigations above, which are the subject of many [5-8].

The closed form solution for the problem at hand has some advantages such as better insight to kinematic behavior of the system, determination of all possible configurations of moving platform for a given input and effect of input errors on output ones.

In this paper, the FPK of a parallel manipulator with a new architecture, which is supposed to be used as a moving mechanism in a flight simulator project, is determined first by using closed form solution. Up to here, dialytic elimination is used for FPK solution. By using this method, the kinematic constraint equations for the system at hand are written and then at each step, one of the unknowns is eliminated from the system of nonlinear equations to obtain the final polynomial equation with one unknown. Thereafter, solving this equation and back substituting to former system of equations, it is possible to determine all unknowns [2,10]. A numerical example is also performed by using dialytic elimination. The results are verified by using other methods such as geometry approach and a numerical method known as polynomial continuation. The kinematics of the manipulator is also studied at position, velocity and acceleration level without deriving the analytical solution of FPK of the system at hand [8].

### CLOSED FORM SOLUTION

The parallel manipulator devised is composed of moving platform (MP) and based platform (BP) that are connected to each other by means of four legs, as depicted in Figure 2.

Each leg contains two links coupled by a prismatic joint. The central leg is connected to MP

by universal joint and fixed to BP. The system has three degrees of freedom (DOF) and three linear actuators are connected to three circumferential legs. It may be noted that the central leg is an idle leg. More details about geometry of the manipulator and its merits as compared with conventional type like Stewart platform has been explained in References 8 and 9. The three independent DOF for the MP are heave  $h$ , i.e., vertical displacement along  $Z$  axis; roll  $\phi$ , i.e., rotation of MP about  $x$  axis and pitch  $\psi$ , i.e., rotation of MP about  $y$  axis, as shown in Figure 2.

The kinematic constraint equations can be written to determine the closed form solution for the FPK of this manipulator. These equations can be expressed in terms of Cartesian space variables, i.e.,  $h$ ,  $\phi$  and  $\psi$  with respect to joint space variables, i.e., the leg lengths  $q_i$  for  $i=1,2,3$ . With reference to Figure 2, the position vector of the upper endpoint of each leg with respect to lower endpoint can be expressed in reference frame attached to BP at its centroid  $O$ , i.e.,  $X_0Y_0Z_0$  as

$$\mathbf{l}_i = -\mathbf{u}_i + \mathbf{p} + \mathbf{R}\mathbf{r}_i = q_i\mathbf{e}_i \quad i=1,2,3 \quad (1)$$

where  $\mathbf{u}_i$  is position vector of  $a_i$  with respect to  $O$ ,  $\mathbf{e}_i$  is the unit vector along  $a_iA_i$  and  $\mathbf{r}_i$  is position vector of  $A_i$  with respect to  $M$ .

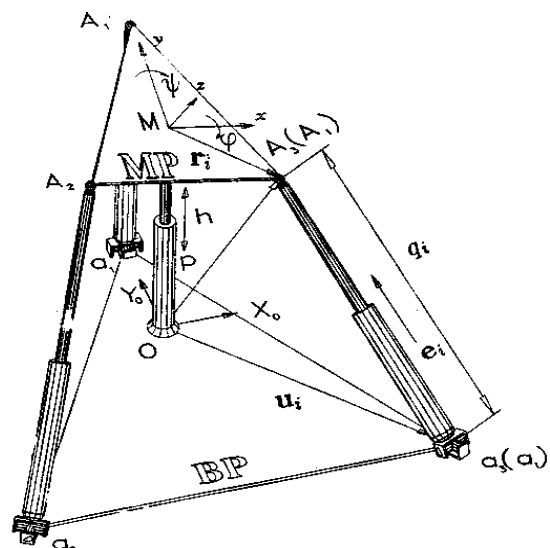


Figure 2. Parallel manipulator with new architecture.

The length of leg  $i$ , i.e.,  $q_i$  can be obtained from Equation 1 as

$$q_i^2 = \mathbf{1}_i^T \mathbf{1}_i = (-\mathbf{u}_i + \mathbf{p} + \mathbf{R}\mathbf{r}_i)^T (-\mathbf{u}_i + \mathbf{p} + \mathbf{R}\mathbf{r}_i) \quad i=1,2,3 \quad (2)$$

where  $\mathbf{R}$  and  $\mathbf{p}$  are the rotation matrix of MP with respect to BP and position vector of upper endpoint of central leg with respect to its lower endpoint, respectively. They are expressed in reference frame as

$$\mathbf{R} = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ \sin\phi \sin\psi & \cos\phi & -\sin\phi \cos\psi \\ -\cos\phi \sin\psi & \sin\phi & \cos\phi \cos\psi \end{bmatrix}$$

$$\mathbf{p} = [0 \quad 0 \quad h]^T \quad (3)$$

Equation 2 contains three kinematics constraints written for legs one, two and three. For simplicity, it is assumed that the length of sides of MP and BP are  $l_{MP} = 2\sqrt{3}b$  and  $l_{BP} = 2\sqrt{3}a$ , respectively. Here,  $a$  is one third of height of equilaterals  $a_1a_2a_3$  and  $b$  is one third of that of  $A_1A_2A_3$ . Moreover, it is noted that the endpoint  $A_1$  of leg 1 is moved on  $Y_0Z_0$  plane. Thus the position vectors  $\mathbf{u}_i$  and  $\mathbf{r}_i$  can be obtained as

$$\mathbf{u}_1 = [0 \quad 2a \quad 0]^T \quad \mathbf{u}_2 = [-\sqrt{3}a \quad -a \quad 0]^T$$

$$\mathbf{u}_3 = [\sqrt{3}a \quad -a \quad 0]^T \quad (4)$$

$$\mathbf{r}_1 = [0 \quad 2b \quad 0]^T \quad \mathbf{r}_2 = [-\sqrt{3}b \quad -b \quad 0]^T$$

$$\mathbf{r}_3 = [\sqrt{3}b \quad -b \quad 0]^T \quad (5)$$

where  $\mathbf{u}_i$  is expressed in reference frame and  $\mathbf{r}_i$  is expressed in frame  $xyz$ , i.e., moving coordinate frame attached to MP at its centroid M.

Substituting Equations 4 and 5 into Equation 2 and expanding them, one obtains

$$4a^2 - 8ab \cos\phi + h^2 + 4bh \sin\phi + 4b^2 = q_1^2 \quad (6)$$

$$4a^2 - 6abc\cos\psi - 2\sqrt{3}abs\sin\phi\sin\psi - 2abc\cos\phi + 4b^2 + h^2 + 2\sqrt{3}bh\cos\phi\sin\psi - 2bh\sin\phi = q_2^2 \quad (7)$$

$$4a^2 - 6abc\cos\psi + 2\sqrt{3}abs\sin\phi\sin\psi - 2abc\cos\phi + 4b^2 + h^2 - 2\sqrt{3}bh\cos\phi\sin\psi - 2bh\sin\phi = q_3^2 \quad (8)$$

Equations 6, 7 and 8 are three equations in terms of three unknowns  $h$ ,  $\phi$  and  $\psi$ . It is readily known from above equations that if  $h^*$ ,  $\phi^*$  and  $\psi^*$  form a set of solution for the above equations then  $-h^*$ ,  $-\phi^*$  and  $-\psi^*$  will also form a set of solution. This will be verified later in numerical example. Expressing the angle  $\psi$  in terms of half angle  $z \equiv \tan(\psi/2)$ , Equations 7 and 8 lead to

$$\alpha z^2 + \beta z + \gamma = 0$$

$$\lambda z^2 + \eta z + \xi = 0 \quad (9)$$

The coefficients of Equations 9 are functions of  $\phi$  and  $h$  and are defined in the Appendix. The resultant of these equations represents the condition under which the two equations have the same solution for  $z$ . This resultant can be determined under the condition that the determinant of Sylvester matrix of Equations 9 vanishes, namely

$$\det \mathbf{S}_1 = 0 \quad (10)$$

where

$$\mathbf{S}_1 = \begin{bmatrix} \alpha & \beta & \gamma & 0 \\ 0 & \alpha & \beta & \gamma \\ \lambda & \eta & \xi & 0 \\ 0 & \lambda & \eta & \xi \end{bmatrix}$$

with the expansion of Equation 10, one leads to an equation that has unknowns  $\phi$  and  $h$ . Thus Equations 6 and 10 lead to two equations that are only functions of  $h$  and  $\phi$ . Rearranging these two equations in terms of  $h$  and expanding the equations obtained, one can derive

$$\delta_2 h^2 + \delta_1 h + \delta_0 = 0 \quad (11)$$

$$\mu_6 h^6 + \mu_5 h^5 + \mu_4 h^4 + \mu_3 h^3 + \mu_2 h^2 + \mu_1 h + \mu_0 = 0$$

The coefficients are in terms of  $\sin\phi$  and  $\cos\phi$  and other known quantities that are defined in the Appendix. Eliminating  $h$  from Equations 11, one can also obtain an equation in terms of only one unknown, namely,  $\phi$ .

The resultant of Equations 11 is obtained upon vanishing the determinant of Sylvester matrix of

two equations as  
 $\det \mathbf{S} = 0$  (12)  
 where  $\mathbf{S}$  is defined as

$$\mathbf{S} = \begin{bmatrix} \mu_6 & \mu_5 & \mu_4 & \mu_3 & \mu_2 & \mu_1 & \mu_0 & 0 \\ 0 & \mu_6 & \mu_5 & \mu_4 & \mu_3 & \mu_2 & \mu_1 & \mu_0 \\ \delta_2 & \delta_1 & \delta_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_2 & \delta_1 & \delta_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_2 & \delta_1 & \delta_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 & \delta_1 & \delta_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_2 & \delta_1 & \delta_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_2 & \delta_1 & \delta_0 \end{bmatrix}$$

Expressing the angle  $\phi$  in terms of half angle  $\phi/2$ , i.e.,  $y = \tan(\phi/2)$ , and substituting into Equation 12, after expansion and simplification, a polynomial equation of order of 24 with respect to  $y$  will be obtained with at most 24 real or complex solutions for  $y$ . The angle  $\phi$  is bound to the interval  $-\pi \leq \phi \leq \pi$  and thus it is uniquely obtained by  $y = \tan(\phi/2)$ . Moreover, after determining  $\phi$ , Equations 11 will have two parts with one unknown,  $h$ , which can be determined by vanishing the first order greatest common divisor (GCD) of the equations. After substituting the values of  $\phi$  and  $h$  into Equations 9, these equations will be a set of two equations in terms of only one unknown  $z$  and they have only one common solution for  $z$  that can be determined upon vanishing the first order GCD of these equations. Thus, it is shown that for each value of  $y$ , there is only one unique solution for  $\phi$ ,  $h$  and  $\psi$ . It may be noted that the GCD of equations in each of the two above-mentioned steps is proved to be of the first order [9]. FPK problem of the manipulator devised has, therefore, at most 24 real and complex solutions.

**Numerical Example** In the manipulator devised, using inverse position kinematics for given values of  $h$ ,  $\phi$  and  $\psi$ , with known geometry, it is possible to determine unique solution for  $q_1$ ,  $q_2$  and  $q_3$ . These values are used

as input for FPK problem to obtain 24 set of solutions. The geometrical parameters for the manipulator are as follows:  $l_{BP} = 2m$  and

$$l_{MP} = 1m \text{ which leads to } a = \frac{1}{\sqrt{3}}m \text{ and}$$

$$b = \frac{1}{2\sqrt{3}}m. \text{ Moreover, using the following values}$$

for the Cartesian space variables:  $h=1m$ ,

$$\phi = -\frac{\pi}{6} \text{ rad and } \psi = -\frac{\pi}{6} \text{ rad and solving the}$$

inverse position kinematics, one obtains

$$q_1 = 0.9667m; q_2 = 1.1060m; q_3 = 1.5420m \text{ (13)}$$

Now, having  $a$  and  $b$  as well as  $q_1$ ,  $q_2$  and  $q_3$  from above equation as input data for FPK and using the method explained in this section, it is possible to obtain 24<sup>th</sup> order polynomial equation. The solution of this equation leads to 24 sets of solutions; twelve sets of them are shown in Table 1 and the other twelve sets are the negatives of the former. It may be noted that there are real and complex solutions which are depicted in Table 1. Each parenthesis in each column of Table 1 consists of the real and imaginary parts of the solution.

## DISCUSSION

The following conclusions can be obtained from the results of the numerical examples described in the previous section:

I- The 24<sup>th</sup> set of the solution that is negative of the first set is the same as the input for the problem.

II- The solutions verify that if  $h^*$ ,  $\phi^*$  and  $\psi^*$  is a set of solution, then  $-h^*$ ,  $-\phi^*$  and  $-\psi^*$  is also a set of solution. Therefore, for  $2m$  ( $m \leq 12$ ) real set of solutions, there are  $m$  configurations under which MP is above BP and the other  $m$  configurations are the ones that MP is under BP. In other words, the former is the mirror image of the latter.

III- All solutions are verified by substituting into kinematics constraint equations, namely, Equations 6, 7 and 8. Therefore, there are no spurious solutions.

TABLE 1. Twelve FPK Solutions of Numerical Example.

No.	$h(m)$	$y = \tan(\phi/2)$	$z = \tan(\psi/2)$
1	(-1.0000,0)	(0.2679,0)	(0.2679,0)
2	(-0.8934,-1.1049)	(1.2952,-0.5902)	(-0.3630,0.9320)
3	(-0.8934,1.1049)	(1.2952,0.5902)	(-0.3623,-0.9320)
4	(-0.7575,0.8231)	(1.1511,0.2327)	(0.4761,0.8092)
5	(-0.7575,-0.8231)	(1.1511,-0.2327)	(0.4761,-0.8092)
6	(-0.7454,0)	(0.6823,0)	(0.4193,0)
7	(-0.3597,-0.5887)	(-0.3222,0.4123)	(-1.5896,4.1832)
8	(-0.3597,0.5887)	(-0.3222,-0.4123)	(-1.5896,-4.1832)
9	(-0.6785,0)	(-0.07967,0)	(1.0669,0)
10	(-0.1567,0)	(0.6190,0)	(1.1960,0)
11	(0,-1.2582)	(0,5.1429)	(0,-5.6852)
12	(0,-1.0882)	(0,1.5123)	(0,0.0694)

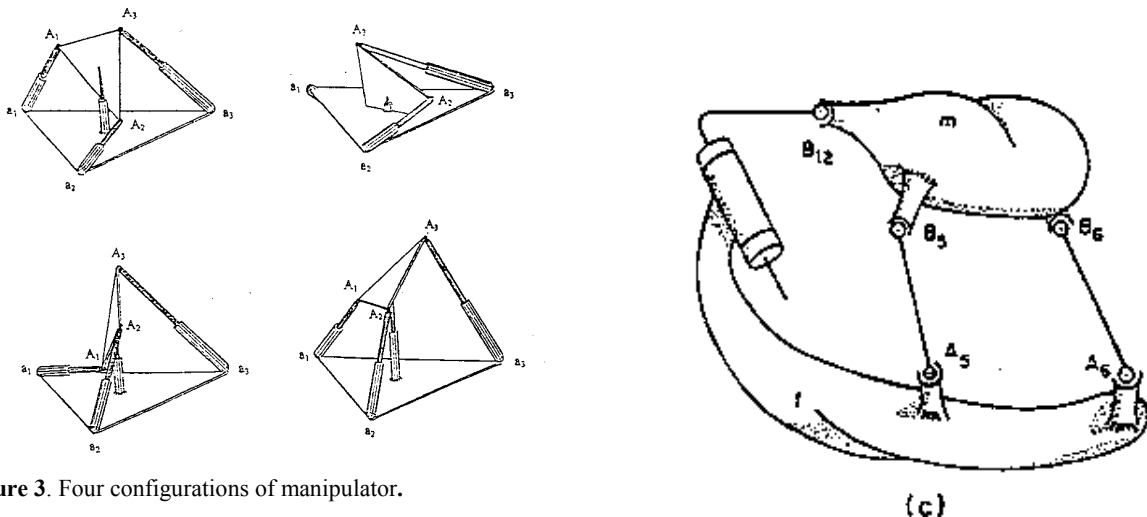


Figure 3. Four configurations of manipulator.

IV- The number of real solutions for the given parameters in this example is eight, which is the number of different configurations of the manipulator. Four configurations where MP is above BP are depicted in Figure 3. It may be noted that it is possible to obtain up to 16 real solutions using different parameters.

### GEOMETRICAL APPROACH

In this section, the maximum number of solutions for FPK of the manipulator devised is studied using synthetic geometry and the method explained

Figure 4. A kinematics connection between frame  $f$  and moveable body  $m$  with feather of 24 and circularity of 12 (with courtesy of Reference 6).

in Reference 6. With reference to the manipulator shown in Figure 2, it is readily known that joint  $A_1$  is located always in  $Y_0Z_0$  plane. Thus, joint  $a_1$  can be considered revolute in equivalent mechanism. Now assume the manipulator is separated at point M, i.e., the endpoint of central leg and compare it with Figure 11(c) of Reference 6, which is depicted

again in Figure 4.

Comparing these two mechanisms, it is found that they are the same and thus the point M of MP develops a surface in reference frame with the order of 24 and circularity of 12. On the other hand, the endpoint of central leg M is moved on a vertical line of reference frame. This line is a first order curve with circularity zero. Therefore, regarding to continuity of the manipulator and referring to Bezout theorem illustrated in Reference 6, maximum number of common real point between the former curve and the latter surface is determined to be as

$$n_1 n_2 - 2p_1 p_2 = 24 \times 1 - 2 \times 12 \times 0 = 24 \quad (14)$$

where  $n_i$  and  $p_i$  are the order and circularity of surface  $i$ , respectively. Thus, the feather of the manipulator is 24 and maximum number of different configurations for the system is determined to be 24.

### POLYNOMIAL CONTINUATION

As it is shown in the previous section, the FPK problem leads to a system of nonlinear multivariable equations. Here, polynomial continuation, which is a numerical technique, is used to solve the FPK problem of the manipulator devised [7]. Then it is shown that there are at most 24 solutions for the FPK problem.

Assuming  $\sin\phi = x$ ,  $\cos\phi = y$ ,  $\sin\psi = m$ , and  $\cos\psi = n$ , the kinematics constraints, namely, Equations 6, 7 and 8 can be simplified in light of the two constraints dictated to the system because of the above relations as

$$\begin{aligned} 4a^2 - 8aby + 4b^2 + h^2 + 4bhx - q_1^2 &= 0 \\ 4a^2 - 6abn + 4b^2 - 2\sqrt{3}abxm - 2aby \\ + h^2 + 2\sqrt{3}bhym - 2bhx - q_2^2 &= 0 \\ 4a^2 - 6abn + 4b^2 + 2\sqrt{3}abxm - 2aby \\ + h^2 - 2\sqrt{3}bhym - 2bhx - q_3^2 &= 0 \\ x^2 + y^2 &= 1 \\ m^2 + n^2 &= 1 \end{aligned} \quad (15)$$

Substituting  $h^2$  from the first equation of set of Equations 15 into the second and third ones and rearranging the result thus obtained leads to

$$\begin{aligned} 4a^2 - 8aby + 4b^2 + h^2 + 4bhx - q_1^2 &= 0 \\ -6abn - 2\sqrt{3}abxm + 6aby - 6bhx \\ + 2\sqrt{3}bhym + q_1^2 - q_2^2 &= 0 \\ -6abn + 2\sqrt{3}abxm + 6aby - 6bhx \\ - 2\sqrt{3}bhym + q_1^2 - q_3^2 &= 0 \\ x^2 + y^2 &= 1 \\ m^2 + n^2 &= 1 \end{aligned} \quad (16)$$

**Upper Limit for Number of Solutions** The first step is to obtain a distribution for variables of Equations 16 leading to a number, as small as possible, known as multihomogeneous Bezout number (MBN). The MBN is the upper bound on the number of finite solutions to the system of Equations 16. For obtaining MBN, variables are classified into three groups, namely,

$$I = \{h\}; \quad II = \{x, y\}; \quad III = \{m, n\} \quad (17)$$

The order of each equation of the system of the Equations 16 versus the variables group, i.e., Equation 17 is shown in Table 2.

The MBN can be determined by forming a polynomial  $a^* = \prod_{i=1, \dots, p} (\sum_{j=1, \dots, q} d_{ij} \alpha_j)$  using the data given in Table 2. Here,  $p=5$  and  $q=3$  are the number of equations and number of variable groups for the system of equations of this manipulator, respectively. Moreover,  $d_{ij}$  is the rank of equation  $i$  in terms of variable group  $j$  that is obtained in the light of Table 2. The number of  $\alpha_j$  is equal to  $q$ , i.e., the number of variable groups. Thus  $a^*$  can be obtained as

$$a^* = (2\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + \alpha_3)^2(2\alpha_2)(2\alpha_3) \quad (18)$$

The coefficient of the term  $\prod_{j=1,2,3} (\alpha_j^{m_j})$  in  $a^*$  is MBN. Here,  $\alpha_j^{m_j}$  is a dummy variable as defined in  $a^*$  and  $m_j$  is the number of variables in group  $j$ . This term is obtained by expanding  $a^*$ , in the light of Equation 18, which is equal to  $24\alpha_1\alpha_2^2\alpha_3^2$ . The coefficient of this term is 24 and thus the upper bound on the number of solution to the system of Equations 16 is 24.

TABLE 2. The Order of the System of Equations 16 Versus the Variables Group.

Group	1st equation	2nd equation	3rd equation	4th equation	5th equation
I	2	1	1	0	0
II	1	1	1	2	0
III	0	1	1	0	2

**All Solutions** A start system is introduced to calculate all solutions for the system of Equations 16. The characteristics of the start system have been mentioned in Reference 7. To begin solving the system devised, let's start with the following equations:

$$\begin{aligned}
 (h - 5 - 12i)(h + 1 - 4.3i)(y - 13 - 5i) &= 0 \\
 (h + 10 + 7i)(y + 30 + i)(n - 100 + 23i) &= 0 \\
 (h - 0.4 + 3i)(y - 40 + 20i)(n + 2 - 9i) &= 0 \quad (19) \\
 x^2 - 1 &= 0 \\
 m^2 - 1 &= 0
 \end{aligned}$$

The above set of polynomial equations has the same structure as the main system, i.e., Equations 16.

These equations have been chosen such that their solutions are distinct and easy to obtain. The numbers used in Equations 16 are selected arbitrary such that these equations satisfy all required conditions and have exactly 24 solutions [9]. Next, using the homotopy as follows makes the transformation of the coefficients of the start system into main system:

$$H_i(t) = (1-t)c_i(f_i)_{start} + t(f_i)_{main} \quad i = 1, \dots, 5 \quad (20)$$

Where  $c_i$  is arbitrary constants chosen as

$$\begin{aligned}
 c_1 &= 20 + 3i, \quad c_2 = -13 + 25i, \quad c_3 = 19 + 7i \\
 c_4 &= 22 - 11.1i, \quad c_5 = -33 - 8.5i \quad (21)
 \end{aligned}$$

$(f_i)_{start}$  and  $(f_i)_{main}$  are the  $i^{th}$  equations of start and main system, respectively.

Equation 20 is solved by using Newton's Raphson method 24 times for each set of solution of start system at  $t=0$  while  $t$  is varied from zero to one in the incremental steps. At each incremental step, the solution of the first step is used as initial condition for the next step. This method is leading to the final solution of equations of the main system, i.e. Equations 16. It is only sufficient to show that there exists one set of input values in which 24 distinct solutions are obtained by executing the above algorithm.

To this end and to verify the results of the previous section, a numerical example is also performed in this section using the same input data as in above-mentioned section. The results are 24 set of solutions that are in a very good agreement with the ones in the previous section. Therefore, the FPK problem is solved using polynomial continuation and is verified that it has 24 solutions.

## CONCLUSIONS

In this paper, the closed form solution for the forward position kinematics of a parallel manipulator with new architecture was derived using dialytic elimination. Using the kinematics constraints of the system and eliminating the variables in two steps, a linear polynomial equation in terms of one unknown, namely, closed form solution was obtained. It was concluded from the final equation that there were at most 24 solutions for the FPK of the manipulator. One numerical

example was also performed to show the above procedure and to obtain all possible solutions as well. The results obtained from this method were verified using other method such as a numerical method known as polynomial continuation. The maximum number of solutions has been verified using geometry approach.

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### APPENDIX

The coefficients of Equations 9 are

$$\alpha = -2abc\cos\phi + h^2 + 6ab + 4b^2 - 2bhsin\phi + 4a^2 - q_2^2$$

$$\beta = -4\sqrt{3}b(asin\phi - h\cos\phi)$$

$$\gamma = -2abc\cos\phi + h^2 - 6ab + 4b^2 - 2bhsin\phi + 4a^2 - q_2^2$$

$$\lambda = -2abc\cos\phi + h^2 + 6ab + 4b^2 - 2bhsin\phi + 4a^2 - q_3^2$$

$$\eta = 4\sqrt{3}b(asin\phi - h\cos\phi)$$

$$\xi = -2abc\cos\phi + h^2 - 6ab + 4b^2 - 2bhsin\phi + 4a^2 - q_3^2$$

The coefficients of first equation of Equations 11 are

$$\delta_2 = 1$$

$$\delta_1 = 4b\sin\phi$$

$$\delta_0 = 4a^2 + 4b^2 - 8ab\cos\phi - q_1^2$$

The second equation of set of Equations 11 is given in Table 3 where  $k_1 = 4(a^2 + b^2) + 6ab$  and  $k_2 = 4(a^2 + b^2) - 6ab$ .

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TABLE 3. The Second Equation of Set of Equations 11.

$$192b^2\cos(\phi)^2h^6 - 384(2b\cos(\phi) + a)b^2\sin(\phi)\cos(\phi)h^5 + 192\left\{ \begin{aligned} &4b^2\sin(\phi)^2\cos(\phi)^2 + 8ab\sin(\phi)^2\cos(\phi) - 4ab\cos(\phi)^3 + a^2\sin(\phi)^2 \\ &- q_2^2\cos(\phi)^2 - \cos(\phi)^2q_3^2 - \cos(\phi)^2k_2 + \cos(\phi)^2k_1 \end{aligned} \right\} b^2h^4 + 384\left\{ \begin{aligned} &-4ab^2\sin(\phi)^2\cos(\phi) + 4ab^2\cos(\phi)^3 - 2a^2b\sin(\phi)^2 + 4a^2b\cos(\phi)^2 \\ &+ bq_2^2\cos(\phi)^2 + bq_3^2\cos(\phi)^2 + aq_2^2\cos(\phi) + aq_3^2\cos(\phi) - bq_2^2k_2 \\ &- bq_3^2k_1 - a\cos(\phi)k_2 - a\cos(\phi)k_1 \end{aligned} \right\} b^2\sin(\phi)h^3 - 48\left\{ -16a^2b^2\sin(\phi)^4 \right. \\ \left. + 64a^2b^2\sin(\phi)^2\cos(\phi)^2 - 16a^2b^2\cos(\phi)^4 + 16a^3b\sin(\phi)^2\cos(\phi) \right. \\ \left. + 16abq_2^2\sin(\phi)^2\cos(\phi) - 8abq_2^2\cos(\phi)^3 + 16ab\sin(\phi)^2\cos(\phi)q_3^2 \right. \\ \left. - 8ab\cos(\phi)^3q_3^2 + 4a^2q_2^2\sin(\phi)^2 + 4a^2\sin(\phi)^2q_3^2 - 16ab\sin(\phi)^2\cos(\phi)k_2 \right. \\ \left. - 16ab\sin(\phi)^2\cos(\phi)k_1 + 8ab\cos(\phi)^3k_2 + 8ab\cos(\phi)^3k_1 - q_2^4\cos(\phi)^2 \right. \\ \left. - 2q_2^2\cos(\phi)^2q_3^2 - \cos(\phi)^2q_3^4 - 4a^2\sin(\phi)^2k_2 - 4a^2\sin(\phi)^2k_1 \right. \\ \left. + 2q_2^2\cos(\phi)^2k_2 + 2q_2^2\cos(\phi)^2k_1 + 2\cos(\phi)^2q_3^2k_2 + 2\cos(\phi)^2q_3^2k_1 \right. \\ \left. - 4\cos(\phi)^2k_2k_1 \right\} b^2h^2 - 96\left\{ -16a^2b^2\sin(\phi)^2\cos(\phi) + 16a^2b^2\cos(\phi)^3 \right. \\ \left. - 4abq_2^2\sin(\phi)^2 + 8abq_2^2\cos(\phi)^2 - 4ab\sin(\phi)^2q_3^2 + 8ab\cos(\phi)^2q_3^2 \right. \\ \left. + 4ab\sin(\phi)^2k_2 + 4ab\sin(\phi)^2k_1 - 8ab\cos(\phi)^2k_2 - 8ab\cos(\phi)^2k_1 \right. \\ \left. + q_2^4\cos(\phi) + 2q_2^2\cos(\phi)q_3^2 + \cos(\phi)q_3^4 - 2q_2^2\cos(\phi)k_2 - 2q_2^2\cos(\phi)k_1 \right. \\ \left. - 2\cos(\phi)q_3^2k_2 - 2\cos(\phi)q_3^2k_1 + 4\cos(\phi)k_2k_1 \right\} a^2b^2\sin(\phi)h \\ + 96a^2b^2q_2^2\sin(\phi)^2q_3^2 + 48a^2b^2\sin(\phi)^2q_3^4 - 384a^3b^3\sin(\phi)^2\cos(\phi)k_1 \\ + 48a^2b^2q_2^4\sin(\phi)^2 - 96a^2b^2q_2^2\sin(\phi)^2k_1 - 96a^2b^2\sin(\phi)^2q_3^2k_2 \\ - 96a^2b^2\sin(\phi)^2q_3^2k_1 - 96a^2b^2q_2^2\sin(\phi)^2k_2 + 384a^3b^3q_2^2\sin(\phi)^2\cos(\phi) \\ + 384a^3b^3\sin(\phi)^2\cos(\phi)q_3^2 - 384a^3b^3\sin(\phi)^2\cos(\phi)k_2 + q_2^4k_2^2 \\ - 2q_2^4k_2k_1 + q_2^4k_1^2 - 2q_2^2q_3^2k_2^2 + 4q_2^2q_3^2k_2k_1 - 2q_2^2q_3^2k_1^2 \\ + q_3^4k_2^2 - 2q_3^4k_2k_1 + q_3^4k_1^2 + 768a^4b^4\sin(\phi)^2\cos(\phi)^2 \\ + 192a^2b^2\sin(\phi)^2k_2k_1 = 0$$

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