LOAD CARRYING CAPACITY OF SIMPLY SUPPORTED VARIABLE THICKNESS CIRCULAR PLATES

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Abstract The calculation of the load carrying capacity of variable thickness circular plates subjected to arbitrary rotational symmetric loading is presented. The analysis considers plate materials that obey either the square or the Tresca yield criterion. By using upper and lower bound theorems of limit analysis, corresponding estimations for the load carrying capacity of the plate are obtained. It is shown that these two bounds are identical. Therefore, the obtained solution would represent the exact amount of the load carrying capacity of the variable thickness circular plate. Finally, in order to obtain solutions for some special cases, plates having step changes in thickness and those with linear thickness variation are considered and corresponding results are illustrated.

Key Words Circular Plates, Variable Thickness, Symmetric Loading

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INTRODUCTION

In many practical situations, the correct prediction of the load carrying capacity of structures has been a subject of great importance. However, the traditional elastic design has proved unable to present realistic estimations for the load carrying capacity of structures. This is due to the logic behind elastic design according to which a structure would fail if it yields at some point. Nevertheless, it is clear that some relatively considerable reserved load carrying capacity still exists in a structure that has failed according to the elastic design. That is why the structures that have been manufactured according to elastic design principles are typically too heavy and expensive.

In order to obtain more accurate estimations for the load carrying capacity of structures, limit analysis theorems have been proposed and widely used. This method does not possess the above-mentioned shortcomings of the elastic design and solutions are usually obtained somewhat easily. Application of the limit analysis approach results in the design of lighter structures than those obtained through the elastic design. Lighter structures not only need less constructional material and manufacturing effort (and thus are less expensive) but also are frequently requested, especially in the

aerospace transportation technology.

Limit analysis theorems and some of their applications in the structural analysis were first introduced in 1950's. Among the most fundamental works on this subject references 1-4 are noteworthy. These include plastic behavior analyses of some structures carrying specific types of loadings. Comprehensive reviews of the work done on the limit analysis and design of plates, shells, and other mechanical or structural members were presented in [5] and [6].

So far, little attention has been paid to the analysis of the plastic behavior of plates subjected to some general kinds of loadings. As one of the few exceptions, in [7], the present author considered constant thickness circular plates obeying either the square or Tresca yield criterion and derived exact solutions for the case of rotational symmetric loadings. The plates were assumed to be either simply supported or clamped.

Quite recently, several attempts have been made in order to obtain estimations for the load carrying capacity of constant thickness circular plates obeying the so-called unified yield criterion [8,9]. While in [8] the plate has been assumed to resist a uniformly distributed load, the case of arbitrary rotational symmetric loading has been considered in [9]. It should be noted, however, that in more complex problems, analytical solutions with the unified yield criterion are difficult to obtain. Nevertheless, this criterion has the advantage of numerical implementation because its yield surface does enable linear programming methods.

As further research in the above-mentioned field, in the present paper, the load carrying capacity of simply supported circular plates with general radially varying thickness has been determined. The obtained results are applicable to the analysis of circular plates subjected to arbitrary rotational symmetric loadings provided
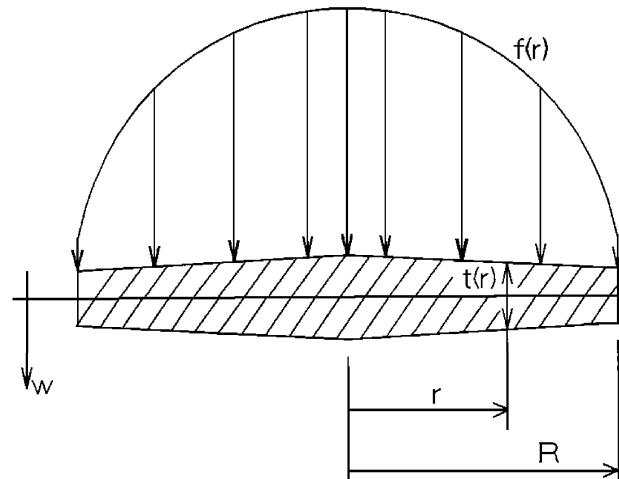

Figure 1. Plate with Rotationally Symmetric Loading and Thickness.

that the plate material obeys either the square or the Tresca yield criterion.

STATEMENT OF THE PROBLEM

Figure 1 demonstrates a variable thickness circular plate of radius R and radially varying thickness $t(r)$ subjected to an arbitrary rotational symmetric loading $f(r)$ per unit area. It is assumed that the plate, in its undeformed shape, has a plane middle surface with respect to which the upper and lower surfaces of the plate are at equal distances.

Rewriting the loading function as $\mu f(r)$, where μ is called the load factor, the main purpose of this paper is to obtain critical values of μ, i.e. μ_{cr}, which cause the collapse mechanism (which is not necessarily unique) to be generated. When μ_{cr} is evaluated, the load carrying capacity of the plate would then become $\mu_{cr}f(r)$.

In the limit analysis, it is commonly assumed that all deflections are negligibly small until the collapse mechanism is fully formed. Accordingly, the material is supposed to be rigid-perfectly plastic. This assumption implies that the sequence of formation of the yield lines prior to collapse is not significant in the results.
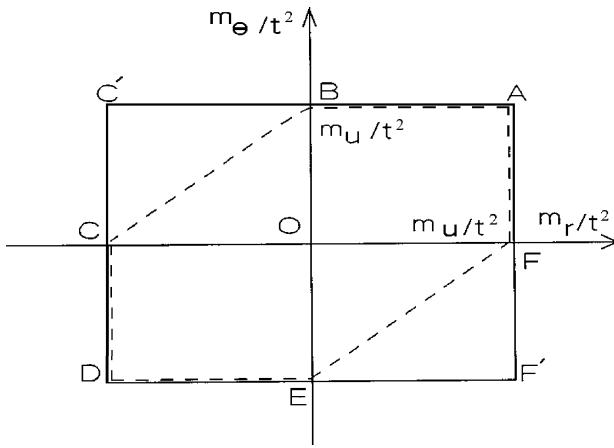
Figure 2. Square (continuous) and Tresca (dashed) yield diagrams.

Furthermore, it is assumed that the material obeys either the square or the Tresca yield criterion, as illustrated in Figure 2. In this figure, m_r, m_θ and m_u stand for the radial, tangential and ultimate (fully plastic) bending moments per unit length, respectively.

It is well known that the ultimate bending moment per unit length for a plate with constant thickness t and yield stress σ_y is,

$$m_u = \frac{1}{4}\sigma_y t^2 \tag{1}$$

For the case of a variable thickness plate, the ultimate bending moment per unit length can be computed in a similar manner. Considering an arbitrary radius r, a circular yield line corresponding to a fully developed plastic region across the plate thickness would be formed only if,

$$m_u(r) = \sigma_y \times \left[1 \times \frac{t(r)}{2}\right] \cdot \left[\frac{t(r)}{2}\right]$$

or,

$$m_u(r) = \frac{1}{4}\sigma_y t^2(r) \tag{2}$$

The first bracket is the area per unit length of the radial yield line on which tensile and compressive components of the fully plastic couple apply. The second bracket is the distance between the resultant forces that generate the couple equal to the ultimate bending moment.

Since $m_u(r)$ is no longer a constant, if the yield criterion is expressed in terms of the bending moment components per unit length, the yield diagram for different points of the plate would not be the same. However, as may be observed from Equation 2, the yield criterion pattern remains unchanged if it is demonstrated in terms of m_r/t^2 and m_θ/t^2. This presentation, being depicted in Figure 2, is used throughout this paper.

CRITICAL LOAD FACTOR FOR SIMPLY SUPPORTED PLATES

For a simply supported plate under transverse loading, it may be examined that the corresponding stress states lie on the line AB in Figure 2, along which the two yield criteria are identical. Hence, the analysis would result in the same outcomes if either of the two criteria is applied.

Considering Koiter's rule [1], the collapse mechanism should be compatible with the assumed yield criterion. It may be easily verified that the frustum collapse mechanism is compatible with the segment AB of the two yield criteria in Figure 2. Hence, regarding these yield criteria, it can be assumed that the frustum collapse mechanism would be generated in the plate.

Figure 3 illustrates the geometric parameters of the frustum shaped collapse mechanism. It should be noted that this mechanism is a generalization of the simple conical one (with $r_f = 0$).

Upper Bound Solution Assuming that the center of the middle surface bears a displacement w_0 during deformation, an investigation of Figure 3 results in,

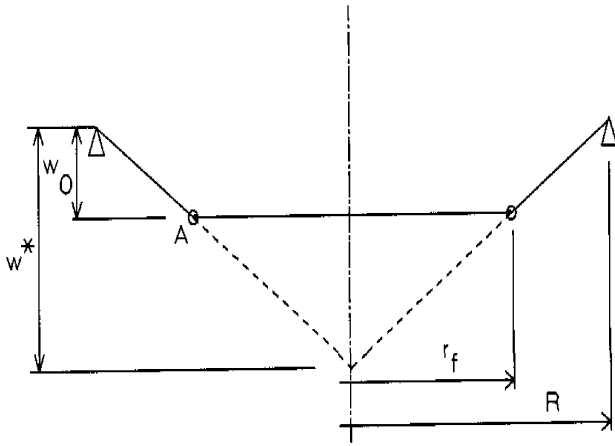$$\frac{w^*}{R} = \frac{w_0}{R - r_f} \tag{3}$$

Figure 3. The frustum collapse mechanism.

or,

$$w_0 = w^*(1 - r_f/R) \qquad (4)$$

where R is the radius of the plate.

The total internal (dissipative) work is the summation of the work done in the inner circular hinge at A, i.e. W_{i1} and that of the conical part of the mechanism, W_{i2} thus,

$$W_i = W_{i1} + W_{i2} \qquad (5)$$

The dissipative work done in the inner circular hinge equals the product of the limit radial bending moment, its length of action, and the jump in the slope of the plate at the hinge. Therefore,

$$W_{i1} = m_u(r_f).(2\pi r_f).\frac{w^*}{R}$$

For the dissipative work done in the conical part of the mechanism, by the application of a hodograph [1], one can write it as the product of the total relative rotation of all infinitesimal radial plate elements which rotate relative to each other along radial yield lines being equal to

$$2\pi(w^*/R)$$

and the ultimate bending moment integrated along a radial yield line. Therefore,

$$W_{i2} = 2\pi \frac{w^*}{R} \cdot \int_{r_f}^{R} m_u(r).dr$$

For the external work done by the applied loading, being equal to the integral on the plate surface of the product of pressure, surface plate element and plate deflection, one obtains,

$$W_e = \int_0^{2\pi} \left[\int_0^R \mu f(r).w(r).r dr \right] d\theta \qquad (6)$$

where w(r) represents the deflection function of the plate. For a frustum collapse mechanism, Equation 6 can be rewritten as,

$$W_e = 2\pi \times$$
$$\left[\int_0^{r_f} \mu f(r) w_0 r dr + \int_{r_f}^{R} \mu f(r) w^*(1 - \frac{r}{R}) r dr \right] \qquad (7)$$

or using Equation 3,

$$W_e = 2\pi \mu w^* \times$$
$$\left[\int_0^{r_f} f(r) r (\frac{R - r_f}{R}) dr + \int_{r_f}^{R} f(r)(1 - \frac{r}{R}) r dr \right] \qquad (8)$$

Considering the virtual work principle and the upper bound theorem of limit analysis, an upper bound estimation for the critical load factor may be obtained by equating Equations 5 and 8. After some simplifications, the following upper bound value for the critical load factor is obtained,

$$\mu_u = \frac{\frac{1}{R} \left[r_f m_u(r_f) + \int_{r_f}^{R} m_u(r) dr \right]}{\int_0^{r_f} f(r) r (\frac{R - r_f}{R}) dr + \int_{r_f}^{R} f(r)(1 - \frac{r}{R}) r dr} \qquad (9)$$

Lower Bound Solution The lower bound theorem of limit analysis is based on the application of the equation of equilibrium of the plate. Furthermore, it requires the introduction of a compatible stress field which never violates the assumed yield criterion. The

equation of equilibrium for circular plates with rotational symmetry geometry, loading, and boundary conditions is as follows [1],

$$(rm_r)' = m_\theta - \int_0^r r^*[\mu f(r^*)]dr^* \qquad (10)$$

Integration of Equation 10 gives,

$$m_r(r) = \frac{1}{r}\int_0^r m_\theta(r^*)dr^* - \frac{\mu}{r}\int_0^r$$

$$\left[\int_0^{r_1} r^* f(r^*)dr^*\right]dr_1 + \frac{c_1}{r} \qquad (11)$$

where, c_1 is the integration constant. Considering the finite requirement at r= 0, this constant vanishes.

For simply supported boundary conditions and either of the two yield criteria, the frustum shape of the collapse mechanism is compatible with a discontinuous m_θ,

$$m_\theta = \begin{cases} m_u(r) & , r_f \leq r \leq R \\ unknown & , 0 \leq r < r_f \end{cases} \qquad (12)$$

In fact, since in the region $r_f \leq r \leq R$ radial yield lines are generated, m_θ is equal to $m_u(r)$. But nothing can be said about the values of m_θ inside the circle $r = r_f$, except that they should satisfy the yield criterion under consideration.

For $r \geq r_f$, the first integral in Equation 11 can be subdivided in two integrals. Therefore, one obtains,

$$m_r(r) = \frac{1}{r}\left[\int_0^{r_f} m_\theta(r^*)dr^* + \int_{r_f}^r m_\theta(r^*)dr^*\right] \quad (13)$$

$$-\frac{\mu}{r}\int_o^r\left[\int_o^{r_1} r^* f(r^*)dr^*\right]dr_1$$

In the first integral, according to Equation 12, m_θ is unknown. However, in order to evaluate this integral, one may calculate Equation 13 at $r = r_f$ which results in,

$$m_u(r_f) = \frac{1}{r_f}\int_o^{r_f} m_\theta(r^*)dr^* - \frac{\mu}{r_f}\int_o^{r_f}$$

$$\left[\int_o^{r_1} r^* f(r^*)dr^*\right]dr_1 \qquad (14)$$

or,

$$\int_o^{r_f} m_\theta(r^*)dr^* = r_f m_u(r_f) + \mu\int_o^{r_f}$$

$$\left[\int_o^{r_1} r^* f(r^*)dr^*\right]dr_1 \qquad (15)$$

Substitution of Equation 15 in Equation 13 gives,

$$m_r(r) = \frac{1}{r}\left\{r_f m_u(r_f) + \mu\int_0^{r_f}\right.$$

$$\left[\int_0^{r_1} r^* f(r^*)dr^*\right]dr_1 + \int_{r_f}^r m_\theta(r^*)dr^*$$

$$-\frac{\mu}{r}\int_0^r\left[\int_0^{r_1} r^* f(r^*)dr^*\right]dr_1 \qquad (16)$$

Finally, the plate boundary conditions may be applied to the obtained formulation. Since the plate is simply supported, the radial bending moment at the plate edge is zero, i.e.,

$$m_r(r=R) = 0 \qquad (17)$$

Using Equations 16 and 17, the following lower bound for the critical load factor may be obtained.

$$\mu_l = \frac{r_f m_u(r_f) + \int_{r_f}^R m_u(r)dr}{\int_{r_f}^R\left[\int_o^r r^* f(r^*)\right]dr} \qquad (18)$$

It may be verified that for the special case of the conical collapse mechanism with $r_f = 0$, Equation 18 reduces to the corresponding one

in [7].

The foregoing derivation reveals that Equation 18 has been obtained through the application of the plate equilibrium Equation 10. However, it can be considered as a complete lower bound solution only if the satisfaction of the corresponding yield criteria and existence of a rigid inner disc is confirmed.

A necessary condition to satisfy the yield criteria, and hence to insure that Equation 18 is a lower bound solution, is that the integral of m_θ over the inner circular disc of the plate should not exceed the plate load carrying capacity in this region. Hence,

$$r_f m_u(r_f) + \mu \int_o^{r_f} \left[\int_o^{r_1} r^* f(r^*) dr^* \right] \times$$
$$dr_1 \leq \int_o^{r_f} m_u(r^*) dr^* \tag{19}$$

Clearly, if the inner circular disc is rigid (corresponding to frustum collapse mechanism) then Condition 19 will be satisfied. That is, it is a necessary condition for rigidity of the inner circular disc.

Searching for an Exact Solution After calculating upper and lower bound estimations for the critical load factor as stated in Equations 9 and 18 respectively, it is interesting to check if these results can be equal for some specific kind of loading.

Applying integration by parts to the denominator of Equation 18 results in,

$$\int_{r_f}^R \left[\int_0^r r^* f(r^*) dr^* \right] dr =$$
$$\left[r \cdot \int_0^r r^* f(r^*) dr^* \right]_{r_f}^R - \int_{r_f}^R r^2 f(r) dr \tag{20}$$

which can be re-written as,

$$\int_{r_f}^R \left[\int_0^r r^* f(r^*) dr^* \right] dr = [(R - r_f) + r_f] \times$$
$$\left[\int_0^{r_f} rf(r) dr + \int_{r_f}^R rf(r) dr \right]$$
$$- r_f \int_0^{r_f} rf(r) dr - \int_{r_f}^R r^2 f(r) dr$$

or finally,

$$\int_{r_f}^R \left[\int_0^r r^* f(r^*) dr^* \right] dr = (R - r_f) \int_0^{r_f} \tag{21}$$
$$rf(r) dr + \int_{r_f}^R (R - r) rf(r) dr$$

Comparing Equations 9 and 18 and applying the above equation reveal that the lower and upper bound estimations obtained for the critical load factor are identical.

Since for the frustum collapse mechanism, the upper and lower bound estimations obtained for the critical load factor have been proved to be equal, it may be concluded that for simply supported circular plates made of materials that obey either the square or the Tresca yield criteria and subjected to arbitrary rotational symmetric loadings this mechanism is exact. The exact value of the collapse load factor can then be obtained from either Equation 9 or 18. Evidently, the solution would be feasible only if the adopted yield condition is satisfied all over the plate.

CASE STUDIES

Stepped Plate As a special case, a stepped circular plate, shown in Figure 4, under uniform pressure p and with the following values of ultimate bending moment is considered:

$$m_u(r) = \begin{cases} kM & , 0 \leq r < \lambda R \\ M & , \lambda R \leq r \leq R \end{cases} \tag{22}$$
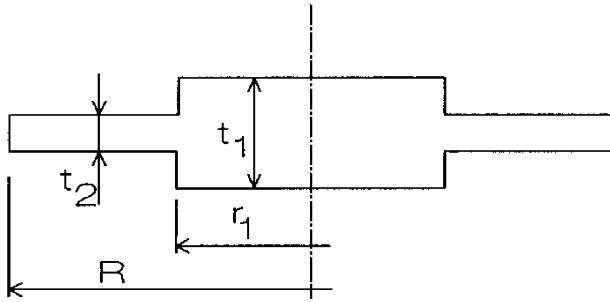
Figure 4. Stepped plate geometry.

Assuming $\lambda = \dfrac{r_1}{R}$, substitution of the loading and thickness functions in the upper bound estimation of the critical load factor, i.e. Equation 9 results in,

$$\mu_u = \frac{6M}{pR^2(1-\lambda^3)} \tag{23}$$

or,

$$p_u = \frac{6M}{R^2(1-\lambda^3)} \tag{24}$$

Application of the lower bound estimation, i.e. Equation 18 provides the same results. Therefore, the obtained upper and lower bounds of the critical load factor coincide and the exact solution of the problem (for $r_1 = \lambda R$) is obtained.

While this solution has been obtained through the application of the equilibrium and virtual work equations, as stated earlier, to ensure that an exact solution has been obtained, the satisfaction of the yield criteria should be verified. Recalling Condition 19 with strict inequality (for rigidity of a central disc in the plate) and utilizing the information given by Equation 22, result in,

$$\lambda R.M + \mu \int_0^{\lambda R} \left[\int_0^{\hat{r}} r^* p \, dr^* \right] \times \tag{25}$$

$$d\hat{r} < \int_0^{\lambda R} kM.dr^*$$

or,

$$\lambda R.M + \mu p \frac{(\lambda R)^3}{6} < \lambda R.kM \tag{26}$$

Substitution for μ from Equation 23 gives,

$$1 + \frac{\lambda^2}{(1-\lambda^3)} < k \tag{27}$$

Since k is a measure of the inner disk reinforcement, this condition can also be interpreted as the necessary reinforcement at the inner disk so that a frustum collapse mechanism can be viable. Using Equation 1, Equation 27 may be rewritten in terms of plate thickness as follows,

$$1 + \frac{(r_1/R)^2}{1-(r_1/R)^3} < \left(\frac{t_1}{t_2}\right)^2 \tag{28}$$

If Conditions 27 or 28 are not satisfied (e.g. if k < 1 which corresponds to a plate with a less stiff inner region), the inner disk necessarily yields and a conical collapse mechanism may be expected.

Linearly Varying Thickness Plate As a second case, a plate with linear variation of thickness which bears some constant pressure p_0 is considered. The thickness function is assumed to be,

$$t(r) = t_c + \frac{t_s - t_c}{R}.r \tag{29}$$

Where, t_c and t_s are the plate thickness at its center and side. Substitution of this thickness function in Equation 2 for ultimate bending moment results in,

$$m_u(r) = \frac{1}{4}\sigma_y \left[t_c + \frac{t_s - t_c}{R}.r \right]^2 \tag{30}$$

With the constant pressure p_0 and above-mentioned function for the ultimate bending moment, the following value of the critical load factor may be obtained from Equation 18,

$$\mu_{cr} = \frac{3}{2} \cdot \frac{\sigma_y}{p_0} \cdot$$

$$\left[\frac{\frac{2}{3}s^2 r_f^3 + s t_c r_f^2 + R t_c^2 + s R^2 t_c + \frac{1}{3}s^2 R^3}{R^3 - r_f^3} \right] \quad (31)$$

where, parameter s is defined as follows,

$$s = \frac{t_s - t_c}{R} \quad (32)$$

The actual shape of the collapse mechanism can be characterized by finding the r_f value which minimizes the quoted critical load factor. Differentiation with respect to r_f results in the

$$\left[s t_c r_f^3 + 3(s^2 R^3 + s R^2 t_c + R t_c^2) r_f + 2 s t_c R^3 \right]$$
$$\times r_f = 0 \quad (33)$$

For the case of s > 0, this equation has only the zero root and no other root can be positive and hence it is physically meaningless. Thus, if the thickness of the plate at its side is more than its thickness at the center, the plate can have only the conical collapse mode.

For s < 0, however, positive real roots may exist. These roots can be found by the application of well known equations for the solution of third order algebraic equations. The true root which correctly specifies the collapse mechanism, is the one that corresponds to the minimum (and not a local maximum) value of the critical load factor.

CONCLUSIONS

Load carrying capacity of simply supported circular plates with radially varying thickness and subjected to arbitrary rotational symmetric loadings has been discussed. In this way, lower and upper bound theorems of limit analysis were implemented and corresponding solutions for the critical value of the load were obtained. The plate material was assumed to obey either the square or the Tresca yield criterion. It was shown that the calculated lower and upper bounds coincide to yield the exact solution. The method was illustrated for the cases of a stepped plate and a plate with linear variation of thickness.

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