PREDICTION OF INSTABILITY IN PLANAR ANISOTROPIC SHEET METAL FORMING PROCESSES

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Abstract In this paper instability of planar anisotropic sheet metal during a few forming processes is investigated for the first time. For this reason components of the constitutive tangent tensor for planar anisotropic sheets are developed. By using the above tensor location of necking is predicted. Direction of the shear band is also predicted using the acoustic tensor. A finite element program is prepared based on large deformations of planar anisotropic sheet metals. In this program rotations of principal directions of anisotropy are also taken in to account. Results obtained from the presented model are in good agreement with experimental observations.

Key Words Sheet Metal Forming, Instability, Bifurcation, Planar Anisotropy, Large Deformation, Necking

چکیده در این مقاله پدیده ناپایداری ورق فلزی ناهمسانگرد صفحه ای در تعدادی از فرایندهای شکل دهی ورق برای اولین بار مورد بررسی قرار گرفته است .بدین منظور ابتدا اجزاء تانسور متشکله مماسی وتانسور اکوستیک برای ورق ناهمسانگرد صفحه ای استخراج گردیده است. با استفاده از این دو تانسور محل و جهت گلویی شدن ورق قابل پیش بینی است. یک برنامه اجزاء محدود بر اساس تغییر شکل های بزرگ ورق ناهمسانگرد صفحه ای تهیه گردیده است. در این برنامه چرخش جهت های اصلی ناهمسانگردی در حین تغییر شکل در نظر گرفته شده است. نتایج بدست آمده از مدل مطابقت خوبی با نتایج تجربی دارد.

INTRODUCTION

Several models have been proposed in the past to describe the plastic instability and predict the limit strains of sheet metals. The limit strains may be considered as a characteristics of sheet metals, which express their formability. In other words the limit strains determine the value of maximum useful deformation prior to localized necking and strain concentration. Some times these limit strains are shown by forming limit diagram(F.L.D.), which is plotted for various loading conditions. Several attempts have been made to predict F.L.D., but most of these theories are put forward for proportional loading conditions.

The problem of diffuse necking has been analyzed by Swift [1]. His analysis of instability in tension is based on the condition of maximum load in two principal directions of strain for a biaxial state of stress. Hill developed a mathematical relation for localized necking by considering velocities in the necking zone [2]. According to his theory, the necking zone starts in a direction of the minor strain equal to zero or less than zero. Hence the limitation of the Hill's model is that it does not work in the stretching region where both major and minor strains are positive. Marciniak and Kuczynski proposed a model suitable for stretching region. Their analysis is based on the idea that necking develops from local region of initial nonuniformity [3]. This model is known as M.K. theory. M.K. model is based on two equations of equilibrium of forces normal to the direction of neck and uniform strain along the neck, both for inside and outside region. Another criterion was put forward by Drucker [4] or Hill [5], which states that a necessary condition for all types of instability (bifurcations and lost of uniqueness) is that the second order work produced by loads for any arbitrary kinematically admissible variation of displacement must be zero or negative. In spite of the early works in criticism of Hill's criterion, Banzant [6,7] put forward thermodynamic arguments for the validity of Hill's theory. Bifurcation at limit point is a special case of Hill's general criterion, which can occur when symmetric part of tangent constitutive tensor obtains a zero eigenvalue. Another case of material instability occurs when rate of displacement field of an elasto-plastic material changes abruptly across a narrow zone, known as shear band. This type of instability is some times called discontinuous bifurcation. Theoretical works of Hadammard [8], Mandel [9] , Rice [10] and Rudnicki *and* Rice [11] have greatly enhanced the understanding of formation of discontinuous bifurcation. According to [9] and [10], the necessary and sufficient condition for discontinuous bifurcation is lost of positive definiteness of acoustic tensor. Recently Otteson *and* Runeson [12] have made the eigenvalue spectral analysis of acoustic tensor. They stated that for a material with symmetric constitutive tangent tensor the condition of discontinuous bifurcation is satisfied when the material is no more hardening. This statement is in agreement with Runesson *and* Laresson's work [13] which indicates that an associated elasto plastic material is stable in Hill's sense when the material is hardening.

In the present work based on Runesson's statement limits of strains and directions of shear bands are obtained for a planar anisotropic material. The components of constitutive tangent tensor for planar anisotropic material are developed for the first time. A finite element program is developed for the simulation of large deformations of planar anisotropic sheet metals [14]. In this program rotations of principal directions of anisotropy are also taken in to account. By using the developed constitutive tangent tensor the location of necking is predicted in this program. Direction of shear band is also predicted using the acoustic tensor. Two sheet metal forming processes are simulated. As the first example stretching of a narrow strip by a hemispherical punch is modeled. As the second example stretching of a circular blank by the same punch is simulated. The results obtained from the present model are in good agreement with the experimental results.

GOVERNING EQUATIONS

According to virtual work principle:

$$\int_v \overline{S}\delta(d\overline{E})dv - \int_s \underline{f}.\delta(\underline{u})ds = 0 \qquad (1)$$

where $\overline{S}$ and $d\overline{E}$ are effective stress and effective strain increment respectively. Based on the above principle and the Hill's theory, a nonlinear finite element program has been developed for analysis of large deformations of planar anisotropic sheet metals. Details of this analysis have been given in [14]. Due to large deformation encountered Lagrangian strains have been used. The effective stress and effective strain increment in the 'Hill's theory' for the state of plane stress conditions are defined as follows [15]:

$$\overline{S} = \sqrt{\frac{3}{2}}(\frac{FS_y{}^2 + GS_x{}^2 + H(S_x - S_y)^2}{F+G+H} + \frac{2NS_{xy}{}^2}{F+G+H})^{\frac{1}{2}} \qquad (2)$$

and,

$$d\overline{E}=\sqrt{\frac{2}{3}}(\frac{(F+H)dE_x^{\ 2}+(H+G)dE_y^{\ 2}}{FG+GH+FH}$$

$$+\frac{2HdE_x dE_y}{FG+GH+FH}+\frac{2dE_{xy}}{N})^{\frac{1}{2}} \qquad (3)$$

In the above equations x and y are principal directions of anisotropy. These directions are parallel and transverse to rolling direction respectively before deformation of the sheet. *F,G* and *H* are anisotropic coefficients. Ratios of these coefficients are assumed to be constant during deformations. In these simulations rotations of principal axes of anisotropy are taken in to account. Yang and Kim have stated that the angle between principal directions of anisotropy and principal directions of stains remain unchanged [16]. This rule is used to obtain principal directions of anisotropy at the end of each step. The principal directions of strains are calculated at the end of each step, using Sowerby method[17].

It should be noted that in the above relations the anisotropic coefficients do not need to be known, and only their ratios *F/H, G/H* and *N/H* are sufficient. If the ratio of width strain to thickness strain in simple tension for strips cut in rolling direction, 45 degrees to rolling direction and 90 degree to rolling direction are defined by *R,Q* and *P* respectively, then according to the "Hill's theory" :

$$\frac{F}{H}=\frac{1}{P},$$

$$\frac{G}{H}=\frac{1}{R},$$

$$\frac{N}{H}=\frac{1}{2}(\frac{1}{P}+\frac{1}{R})(2Q+1)$$

Therefore these ratios can be evaluated by three simple tension tests.

CONSTITUTIVE TANGENT TENSOR

According to the Hill's theory, the yield function is defined as following:

$$f(\sigma)=\frac{1}{2}[F(\sigma_y-\sigma_z)^2+G(\sigma_z-\sigma_x)^2+$$

$$H(\sigma_x-\sigma_y)^2+2N(\sigma_{xy})^2] \qquad (4)$$

The elastic and plastic strain increments are defined as:

$$d\varepsilon_{ij}^{\ e}=\frac{d\sigma_{ij}}{2g}-\delta_{ij}\frac{\nu}{E}\sigma_{kk}$$

and,

$$d\varepsilon_{ij}^{\ p}=\frac{\partial f(\sigma)}{\partial\sigma_{ij}}d\lambda$$

But we know that:

$$d\sigma_{kk}=\frac{E}{1-2\nu}d\varepsilon_{kk}^{\ e}$$

In the other hand $d\varepsilon_{kk}^{\ e}=d\varepsilon_{kk}$, because, $d\varepsilon_{kk}^{\ p}=0$. Therefore elastic strain increment is written as:

$$d\varepsilon_{ij}^{\ e}=\frac{d\sigma_{ij}}{2g}-\delta_{ij}\frac{\nu}{1-2\nu}d\varepsilon_{kk}$$

The total strain increment is obtained as follows:

$$d\varepsilon_{ij}=\frac{d\sigma_{ij}}{2g}-\delta_{ij}\frac{\nu}{1-2\nu}d\varepsilon_{kk}+\frac{\partial f(\sigma)}{\partial\sigma_{ij}}d\lambda \qquad (5)$$

From the above relation:

$$d\sigma_{ij}=2g[d\varepsilon_{ij}-\frac{\partial f(\sigma)}{\partial\sigma_{ij}}d\lambda+\delta_{ij}\frac{\nu}{1-2\nu}d\varepsilon_{kk}] \quad (6)$$

but from the Hill's theory:

$$d\lambda=\frac{3d\overline{\sigma}}{2\overline{\sigma}h(F+G+H)} \qquad (7)$$

where h is the slope of the constitutive curve $\bar{\sigma} - \bar{\varepsilon}$ at a certain point, or, $h = \frac{d\bar{\sigma}}{d\bar{\varepsilon}}$. From the definition of effective stress we have :

$$\bar{\sigma}^2 = \frac{3f(\sigma)}{(F+G+H)} \quad .$$

By differentiating the above relation:

$$d\bar{\sigma} = \frac{3}{2\bar{\sigma}(F+G+H)} \frac{\partial f(\sigma)}{\partial \sigma_{ij}} d\sigma_{ij}$$

Substitution of $d\bar{\sigma}$ in Equation 7 gives:

$$d\sigma_{ij} = \frac{4h\bar{\sigma}^2 (F+G+H)^2}{9 \frac{\partial f(\sigma)}{\partial \sigma_{ij}}} d\lambda \qquad (8)$$

Finally by eliminating $d\lambda$ from Relations 6 and 8 we have:

$$d\sigma_{ij} = 2g(d\varepsilon_{ij} - \frac{9}{4} a \frac{\partial f(\sigma)}{\partial \sigma_{ij}} + \delta_{ij} \frac{\nu}{1-2\nu} d\varepsilon_{kk}) \qquad (9)$$

where a is defined as follows:

$$a = \left[\frac{2g\left[d\varepsilon_{mn} + \delta_{mn} \frac{\nu}{1-2\nu} d\varepsilon_{kk} \right] \frac{\partial f(\sigma)}{\partial \sigma_{mn}}}{h\bar{\sigma}^2 (F+G+H)^2 + 4.5g \frac{\partial f(\sigma)}{\partial \sigma_{pq}} \frac{\partial f(\sigma)}{\partial \sigma_{pq}}} \right]$$

Equation 9 can be summarized as follows:

$$d\sigma_{ij} = D_{ijkl} d\varepsilon_{kl}$$

where D_{ijkl} is the constitutive tangent tensor. The above relation can be written in the following form:

$$d\sigma = D d\varepsilon$$

Where in the plane stress condition $d\sigma$ and $d\varepsilon$ are vectors with 3 elements and D is a (3×3) matrix. In this case the elements of $d\sigma$ and $d\varepsilon$ are as follows:

$$d\sigma = \left[d\sigma_x, d\sigma_y, d\sigma_{xy} \right]^T$$

and,

$$d\varepsilon = \left[d\varepsilon_x, d\varepsilon_y, d\varepsilon_{xy} \right]^T .$$

Components of the acoustic tensor can be obtained from the constitutive tangent tensor:

$$Q_{kl} = n_i D_{ijkl} n_j \quad .$$

In the above equation, n is unit normal vector to the shear band direction. Components of constitutive tangent tensor are presented in appendix.

DISCONTINUOUS BIFURCATION CHARACTERISTICS

Let's consider the current state of static equilibrium with continuous displacement u. When the load is increased a discontinuous bifurcation may occur. It is assumed that an additional displacement rate $\lfloor \dot{u} \rfloor$ appears due to a jump across a fixed shear band. According to Hadamard [8], $\lfloor \dot{u} \rfloor$ must be continuous along the shear band, or:

$$\frac{\partial [\dot{u}_i]}{\partial \xi} = 0 \qquad (10)$$

where ξ is a local axis along the shear band. If η is the axis normal to shear band, then we can write:

$$\frac{\partial [\dot{u}_i]}{\partial x_j} = \frac{\partial [\dot{u}_i]}{\partial \xi} \frac{\partial \xi}{\partial x_j} + \frac{\partial [\dot{u}_i]}{\partial \eta} \frac{\partial \eta}{\partial x_j} \qquad (11)$$

But the first term of the right side of the above relation is zero, therefore:

$$\frac{\partial[\dot{u}_i]}{\partial x_j} = \frac{\partial[\dot{u}_i]}{\partial \eta}\frac{\partial \eta}{\partial x_j} \tag{12}$$

Relation 12 can be written as follows:

$$\frac{\partial[\dot{u}_i]}{\partial x_j} = c_i n_j \tag{13}$$

where *n* is the unit vector along the η axis. If two sides of the shear band are clarified by *I* and *II*, then:

$$\dot{\varepsilon}_{ij}^{\ II} = \dot{\varepsilon}_{ij}^{\ I} + [\dot{\varepsilon}_{ij}] \tag{14}$$

where,

$$[\dot{\varepsilon}_{ij}] = \frac{1}{2}(c_i n_j + c_j n_i) \tag{15}$$

Due to equilibrium, the traction rate across the shear band surface must be identical. Then:

$$(\dot{\sigma}^I - \dot{\sigma}^{II})n = 0 \tag{16}$$

or:

$$(D_{ijkl}^{\ I}\dot{\varepsilon}_{kl}^{\ I} - D_{ijkl}^{\ II}\dot{\varepsilon}_{kl}^{\ II})n_j = 0 \tag{17}$$

Suppose that the material at both sides of the surface responds plastically (plastic-plastic bifurcation). Then from continuity consideration:

$$D_{ijkl}^{\ I} = D_{ijkl}^{\ II} \tag{18}$$

Combination of (14), (15), (17) and (18) gives:

$$n_j D_{ijkl} n_k c_l = 0 \tag{19}$$

or:

$$Q_{il} c_l = 0 \tag{20}$$

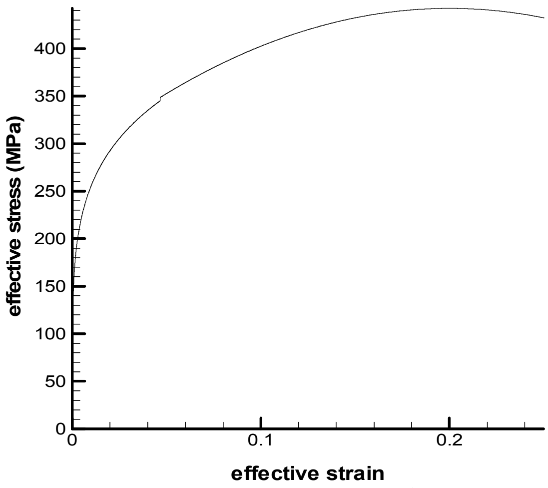


Figure 1 Stress strain curve of the sheet.

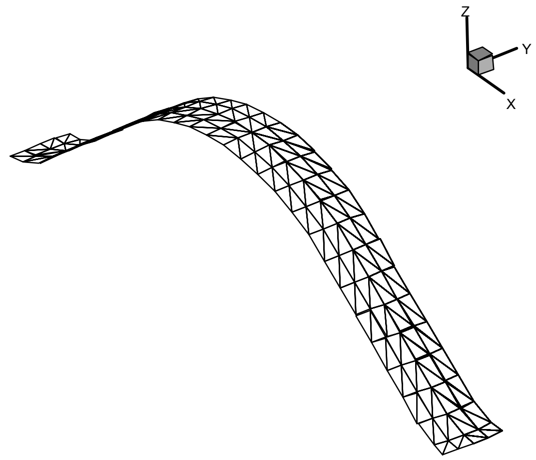


Figure 2. Deformed mesh of the strip.

The non-trivial solution of (20) is possible when acoustic matrix Q_{il} is singular.

RESULTS AND DISCUSSIONS

Two different sheet metal forming processes are

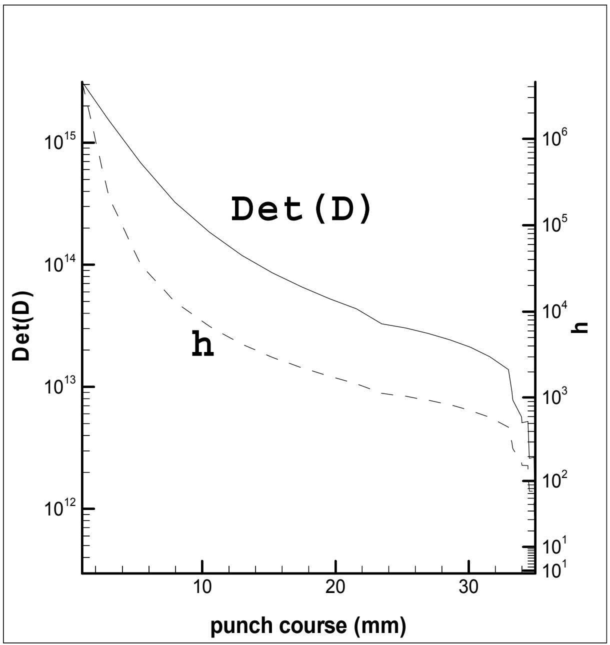


Figure 3. Variations of h and Det (D) at necking element versus the punch course.

modeled. For both of examples the constitutive equation of sheet metal is assumed to be:

$$\bar{\sigma} = 627.420\bar{\varepsilon}^{0.2} \quad \text{Mpa} \qquad \bar{\varepsilon} < 0.0467$$

$$\bar{\sigma} = 0.0282\bar{\varepsilon}^{3} - 3983.62\bar{\varepsilon}^{2} + 1593.44\bar{\varepsilon} + 282.986 \quad \text{Mpa} \qquad \bar{\varepsilon} > 0.0467$$

This relationship is shown in Figure 1.

Strip Stretching As the first example stretching of narrow strips by a hemispherical punch is modeled. In these models coefficient of friction is assumed to be 0.2, width of strips are 12.5 mm, radius of punch is 50 mm and radius of die throat is 53mm. Figure 2 shows the deformed mesh of strip for a case when *R,Q* and *P* are 1, 3 and 1, respectively. Values of *h* and Det (*D*) are calculated at each step, for all of the elements. When *h* or Det (*D*) vanished, necking is occurred. In Figure 3 variations of *h* and determinant of

TABLE 1. Effects of Anisotropy Parameters on the Strip Stretching Process.

R	*Q*	*P*	Punch course at necking (mm)	Angle between the Shear bands (deg)
1	1	1	34.8	69.1
1	1.5	1	35.1	68.7
1	3	1	35.3	68.2
1	4	1	35.8	67.4
1	5	1	36.2	67.0
2	1	2	36.9	78.5
3	1	3	37.4	80.8
4	1	4	38.0	82.5
5	1	5	38.8	83.6
1.9	1.3	1.9	37.4	77.1

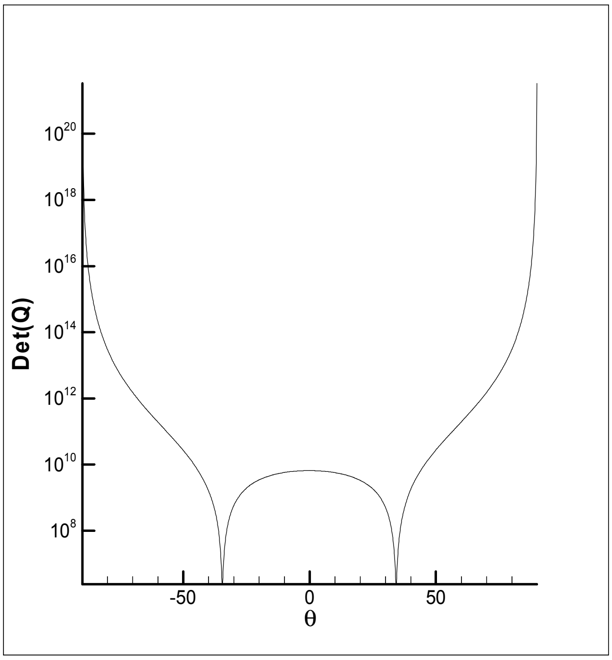


Figure 4. Variations of Det (Q) at necking element in different directions.

matrix *D* for the element at necking zone versus the punch course are shown. This figure shows that *h* and Det (*D*) have similar behavior at the necking zone.

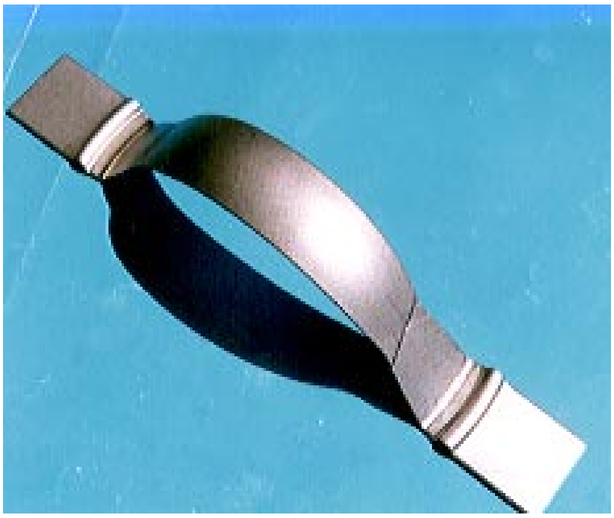


Figure 5. Experimental deformed strip.

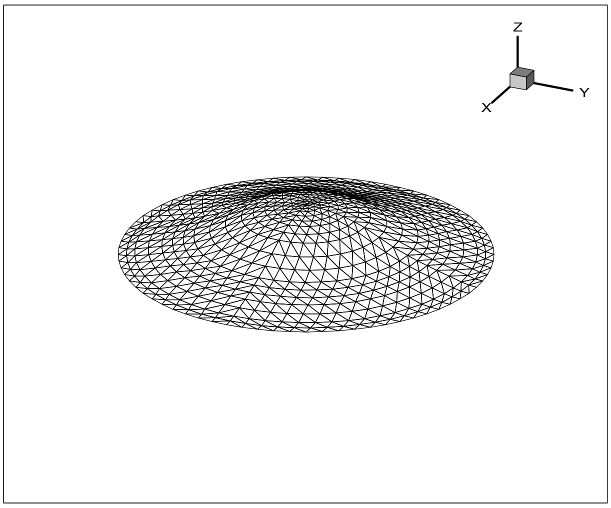


Figure 6 Deformed mesh of circular blank.

The sharp decrement of these values at punch course of almost 34 mm shows that necking has occurred. In Figure 4 variations of determinant of acoustic matrix at the above punch course and necking zone for different directions are shown. In this figure θ is the angle between the normal to shear band and the x-axis, which is the principal direction of anisotropy. It is obvious that values of θ . These values are almost –34 and 34 degrees. It should be noted that in this example the principal direction of anisotropy is longitudinal direction of the strip (rolling direction) during the deformation, because rotations are negligible in this process. The effects of parameters R,Q and P are also studied in this work. In Table 1 effects of these parameters on the punch course at necking and angle between the shear bands are summarized.

It is obvious from this table that in the first case, when R and P are constant, variations of Q does not have significant effect on the punch course at necking and angle between the shear bands. In the second case, when Q is constant variations of R and P is more effective than the first case. The experimental deformed strip is shown in Figure 5. Parameters R,Q and P of the experimented sheet were measured by stretching strips cut in different

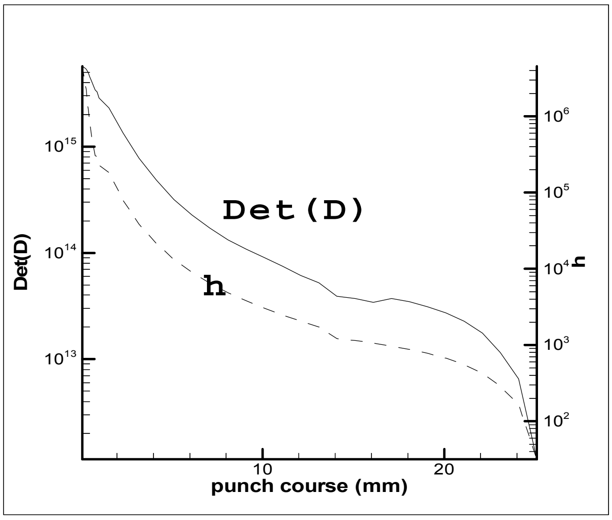
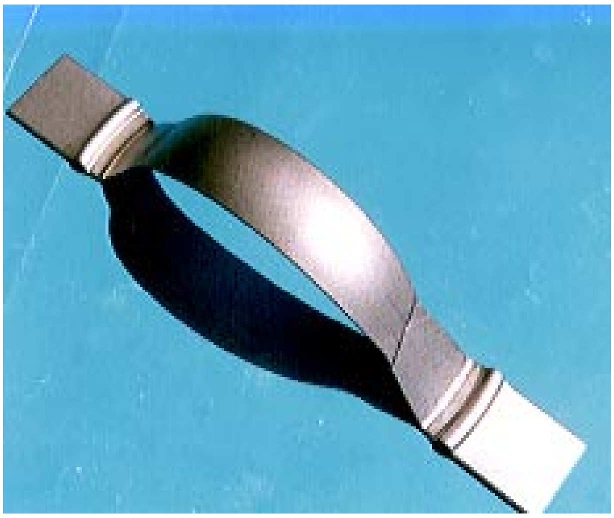


Figure 7. Variations of h and Det (D) at necking.

directions. The average values of these parameters obtained from three experiments, were 1.9,1.3 and 1.9 respectively. The punch course at the onset of necking in this experiment is 36 mm. It is seen from Figure 4 that the angle between the shear bands are about 80 degrees. These results are in good agreement with theoretical values given in the last row of Table 1.

Circular Blank Stretching As the second example a circular blank is stretched by the same



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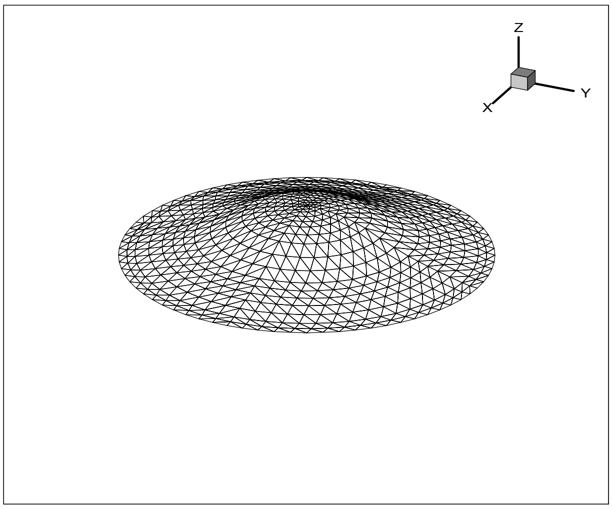

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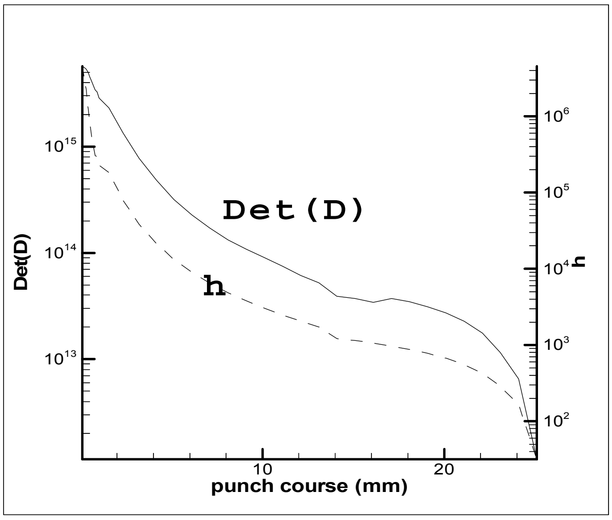
It is obvious from this table that in the first case, when *R* and *P* are constant, variations of *Q* does not have significant effect on the punch course at necking and angle between the shear bands. In the second case, when *Q* is constant variations of *R* and *P* is more effective than the first case. The experimental deformed strip is shown in Figure 5. Parameters *R,Q* and *P* of the experimented sheet were measured by stretching strips cut in different



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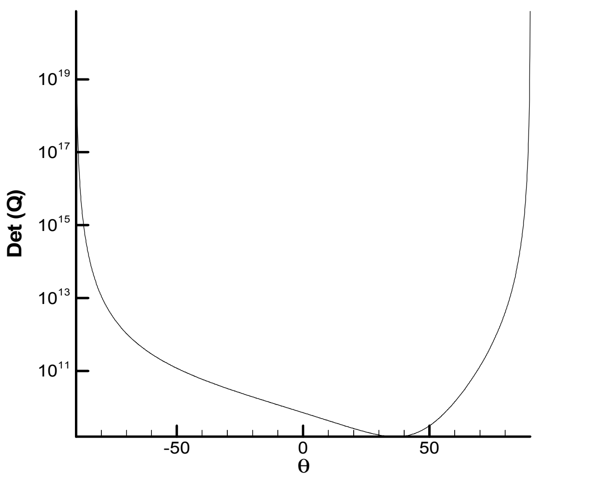

Figure 8. Variations of Det (Q) at necking element verses the punch course.

TABLE 2. Effects of Anisotropy Parameters on Circular Blank Stretching Process.

R	*Q*	*P*	Punch course at necking (mm)	Angle Betwee n mer.line and normal to shear band (deg)
1	1	1	23.5	0.2
1	2	1	24.5	1.5
1	3	1	25.1	14.5
1.9	1.3	1.9	22.4	3.7

punch and die of the first example. Deformed mesh of the blank is shown in Figure 6. For a case when *R,Q* and *P* are 1, 3 and 1 respectively, variations of *h* and Det (*D*) at necking zone, versus the punch course are shown in Figure 7. Variations of these values in this case have the same behavior of the first example. In Figure 8 variations of determinant

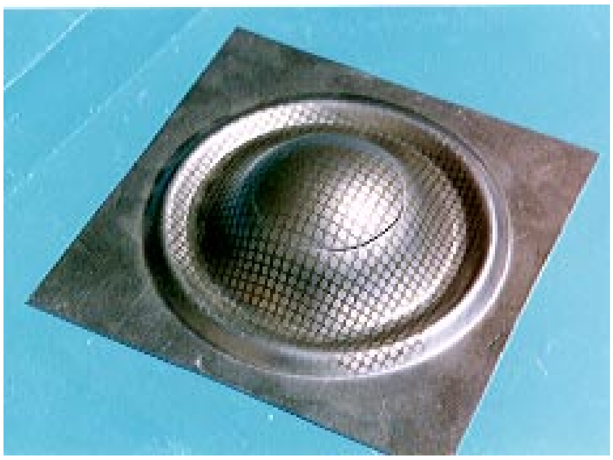

Figure 9. Experimental deformed circular blank.

of acoustic matrix versus θ are shown. It is obvious from this figure that determinant of acoustic matrix vanishes only for one value of θ (36.6 degrees). The angle between meridional line passed through the centeroid of necked element and the rolling direction is 50.9 degrees. The angle between the normal to shear band and the meridional line passed through the centeroid of this element can be calculated from the above values.

This angle is found to be 14.5 degrees for the above case. For a case when *R, Q* and *P* are assumed as 1, 1 and 1, the above angle is almost 0.2 degrees. This example is solved for four cases and the results are summarized in Table 2.

The experimental deformed circular blank is shown in Figure 9. Parameters *R, Q* and *P* were 1.9, 1.3 and 1.9 respectively as in Example 1. It is obvious from this figure that the angle between the normal to shear band and the meridian line is zero. The punch course at the onset of necking is almost 23 mm. Results obtained from the model for the above measured values of *R, Q* and *P* are given at the last row of the Table 2. These results show good agreement with experimental observations.

CONCLUSIONS

A finite element program was prepared for modeling of large deformations of planar anisotropic sheet metals. In this model rotations of principal directions of anisotropy are taken in to account. By developing the constitutive tangent tensor for planar anisotropic sheet metal, and using the Rice theory necking of sheet is predictable in this model. Direction(s) of shear band(s) can also be predicted by this model.

Two examples were solved by the model and the results were compared with the experimental observations. In the first example stretching of a narrow strip is modeled and occurrence of two shear bands is predicted. This prediction is confirmed by the experiment. In the second example stretching of a circular blank is modeled and occurrence of one shear band is predicted. This prediction is also confirmed by the experiment. Both directions of the shear bands and the punch course at the onset of necking obtained from the model are also in good agreement with the experimental observations.

APPENDIX

From Relation 11 we can write:

$$d\sigma_x = 2g\left[\frac{1-\nu}{1-2\nu} - \frac{9}{4}\,\frac{2g[(\frac{1-\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_x})^2 + (\frac{\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \sigma_y}) + (\frac{\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \sigma_z})]}{A}\right]d\varepsilon_x$$

$$+2g\left[\frac{1-\nu}{1-2\nu} - \frac{9}{4}\,\frac{2g[(\frac{\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_x})^2 + (\frac{\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \sigma_y}) + (\frac{1-\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \sigma_z})]}{A}\right]d\varepsilon_z$$

$$+2g\left[-\frac{9}{4}\,\frac{2g\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \sigma_{xy}}}{A}\right]d\varepsilon_{xy} \qquad \text{(A1)}$$

where A is defined as follow:

$$A = h\bar{\sigma}^2(F+G+H)^2 + 4.5g[(\frac{\partial f}{\partial \sigma_x})^2 + (\frac{\partial f}{\partial \sigma_y})^2 + (\frac{\partial f}{\partial \sigma_z})^2 + (\frac{\partial f}{\partial \sigma_{xy}})^2]$$

Permutation of x and y in the Relation A1 gives $d\sigma_y$. But for $d\sigma_{xy}$ from Relation 11 we have:

$$d\sigma_{xy} = 2g\left[-\frac{9}{4}\frac{2g[(\frac{1-\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \sigma_{xy}}) + (\frac{\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_y}\frac{\partial f}{\partial \sigma_{xy}}) + (\frac{\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_z}\frac{\partial f}{\partial \sigma_{xy}})]}{A} \right] d\varepsilon_x$$

$$+ 2g\left[-\frac{9}{4}\frac{2g[(\frac{1-\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_y}\frac{\partial f}{\partial \sigma_{xy}}) + (\frac{\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \sigma_{xy}}) + (\frac{\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_z}\frac{\partial f}{\partial \sigma_{xy}})]}{A} \right] d\varepsilon_y$$

$$+ 2g\left[-\frac{9}{4}\frac{2g[(\frac{1-\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_z}\frac{\partial f}{\partial \sigma_{xy}}) + (\frac{\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_x}\frac{\partial f}{\partial \sigma_{xy}}) + (\frac{\nu}{1-2\nu})(\frac{\partial f}{\partial \sigma_z}\frac{\partial f}{\partial \sigma_{xy}})]}{A} \right] d\varepsilon_z$$

$$+ 2g\left[1-\frac{9}{4}\frac{2g(\frac{\partial f}{\partial \sigma_{xy}})^2}{A} \right] d\varepsilon_{xy} \qquad \text{(A2)}$$

For plane stress condition $d\varepsilon_z$ can be eliminated . The above relations can be summarized as follows:

$$d\sigma_x = A_{xx}d\varepsilon_x + A_{xy}d\varepsilon_y + A_{xz}d\varepsilon_z + A_{xxy}d\varepsilon_{xy} \qquad \text{(A3)}$$

$$d\sigma_y = A_{yx}d\varepsilon_x + A_{yy}d\varepsilon_y + A_{yz}d\varepsilon_z + A_{yxy}d\varepsilon_{xy} \qquad \text{(A4)}$$

$$d\sigma_{xy} = B_{xyx}d\varepsilon_x + B_{xyy}d\varepsilon_y + B_{xyz}d\varepsilon_z + B_{xyxy}d\varepsilon_{xy} \qquad \text{(A5)}$$

In the other hand we know that for plane stress condition, $d\sigma_z = 0$. Therefore by noting that sum of the plastic strain increments is zero, we have:

$$d\sigma_x + d\sigma_y = \frac{E}{1-2\nu}(d\varepsilon_x + d\varepsilon_y + d\varepsilon_z) \qquad \text{(A6)}$$

Substitution of (A3) and (A4) in (A6) gives:

$$d\varepsilon_z = \frac{\frac{1-2\nu}{E}\left[(A_{xx}+A_{yx})d\varepsilon_x + (A_{xy}+A_{yy})d\varepsilon_y + (A_{xxy}+A_{yxy})d\varepsilon_{xy}\right] - d\varepsilon_x - d\varepsilon_y}{1-\frac{1-2\nu}{E}(A_{xz}+A_{yz})} \quad \text{(A7)}$$

Relation A7 is can be written as folows:

$$d\varepsilon_z = C_x d\varepsilon_x + C_y d\varepsilon_y + C_{xy} d\varepsilon_{xy} \quad \text{(A8)}$$

Finally, substitution of (A8) in (A3),(A4) and(A5) gives:

$$d\sigma_x = (A_{xx} + C_x A_{xz})d\varepsilon_x + (A_{xy} + C_y A_{xz})d\varepsilon_y + (A_{xxy} + C_{xy} A_{xz})d\varepsilon_{xy} \quad \text{(A9)}$$

$$d\sigma_y = (A_{yx} + C_x A_{yz})d\varepsilon_x + (A_{yy} + C_y A_{yz})d\varepsilon_y + (A_{yxy} + C_{xy} A_{yz})d\varepsilon_{xy} \quad \text{(A10)}$$

$$d\sigma_{xy} = (B_{xyx} + C_x B_{xyz})d\varepsilon_x + (B_{xyy} + C_y B_{xyz})d\varepsilon_y + (B_{xyxy} + C_{xy} B_{xyz})d\varepsilon_{xy} \quad \text{(A11)}$$

The above relations can be written as folow:

$$d\sigma = D d\varepsilon \quad \text{(A12)}$$

It can be simply proved that matrix *D* is symmetric.

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