SIMULATING A RENEWAL PROCESS WITH GAMMA DISTRIBUTED TIMES

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Abstract When the times between renewals in a renewal process are not exponentially distributed, simulation can become a viable method of analysis. The renewal function is estimated through simulation for a renewal process simulation for a renewal process with gamma distributed renewal times and the shape parameter $\alpha > 1$. Gamma random deviates will be generated by means of the so called Acceptance-Rejection mothod. In order to reduce the variance of the point estimator, the idea of antithetic variates will be incorporated in the sampling process. It is shown that such sampling scheme is capable of reducing the variance of the point estimator. Finally, an algorithm is developed and along with the experimental results is verified.

Key Words Simulation, Variance Reduction, Rejection Method

چکیده به منظور تحلیل یک فرآیند تجدید که پس از گذشت مدتهایی با توزیع غیر نمایی منفی، جدید می شود، شبیه سازی می تواند ابزار مفیدی باشد در این مقاله، مدت زمان مورد اشاره از توزیع گاما با پارامتر شکل بزرگتر از ۱ برخوردارند و از طریق شبیه سازی تابع جدید برآورد می شود. مقادیر تصادفی گاما با استفاده از روش رد و *قبول*، تولید می شوند. به منظور کاستن از واریانس براوردکنندهٔ نقطه ای، از روش مقادیر متضاد نیز در نمونه گیری استفاده می شود. **شرایط و امکان** کاستن از واریانس براوردکنندهٔ نقطه ای امید ریاضی تعداد موارد تجدید، به اثبات می رسد و در پایان **الگوریتمی** در زمینهٔ اجرای روش پیشنهادی ارائه می شود و نتایج تئوری با مثالهای تجربی، مورد مقایسه قرار می گیرد.

INTRODUCTION

Discrete event simulation is an increasingly used method of analysis which requires an enormous amount of computer time for generating representative outputs. Some methods exist for increasing output precision or reducing simulation run time. These methods are generally refered to as the Variance-Reduction Techniques (VRT). Law and Kelton[1] provide a thorough discussion of the general types of VRT's that have been successfuly applied to a wide variety of simulations. There exist four VRT's which are applicable to simulating a single system. Antithetic variate method (AV) is one of these techniques which is effective and easy to implement [1]. Due to its desirable properties, AV is the only VRT which has been incorporated in GPSS/H (See Schriber[2]) while the other types of VRT's are yet to be featured in any simulation language.

Since the random variates are monotonic functions of the random numbers, AV is aided by the inverse transformation method [3]. In fact, if the random deviates are generated through a method other than the inverse transformation, the results of applying the antithetic variates are at best uncertain in the sense that nothing can be said about the induced correlation between pairs of observations.

One of the important probability distribution functions (pdf) is the gamma pdf which is defined as follows:

$$f_x(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, \quad 0 \le x, \quad 0 < \alpha, \beta \tag{1}$$

where α is known as the shape parameter and β is known as the scale parameter. Many random phenomena in a wide variety of applications behave as a gamma random variable.

Generating the gamma random deviates in simulation, under circumstanses that α is non-integer, is limited to the rejection methods or the iterative schemes [4]. The rejection method proposed by Fishman[5] uses a negative exponential pdf as the "enveloping function" for the case of $\alpha > 1$.

The iterative schemes (see Phillips[6]) are typically rather long and time consuming on computer [4].

Due to its nature, the rejection method so far has not positively coexisted in an environment with the technique of antithetic variates. What this paper is trying to achieve is laying down the foundation for such a coexistence. As such, a two parameter gamma pdf with $\alpha > 1$ whose deviates are generated through the rejection method is chosen and a method is proposed which allows the simultaneous application of complementary AV. In order to demonstrate the merits of the algorithm, approximation of the renewal function in a replacement process is chosen as a case problem.

A SINGLE COMPONENT REPLACEMENT PROCESS

A replacement process consists of a series of regeneration times with the following characteristics:

1. times between regenerations form a series of independent and identically distributed (iid) random variables.

2. the expected value of a function of the process immediately after regeneration is independent of the past history of the process.

In a replacement process once a component fails, the system returns to the down condition. Simulation of a single component replacement process depends on the number of replacements required for the operation of the system for **t** time units, N(t), the residual life of the system's component, $R(S_1)$, and the residual lives of the spares, $R(S_2)$, $R(S_3)$,... .The cumulative distribution function (CDF) of the residual life can be expressed as follows:

$$P\{R(s) \leq x\} = [F(x+s) - F(s)][1 - F(s)]^{-1},\ 0 \leq s, x \tag{2}$$

where F(.) represents the CDF of the single component system's life. In fact, it is assumed that the ith spare has s_i hours of accumulated life (i = 1,2,...) at installation, where s_i is not necessarily zero. The relation between N (t) and R (s_i) can be expressed as

$$P\{N(t) = n\} = P\{\sum_{i=1}^{n} R(s_i) \leq t \cap \sum_{i=1}^{n+1} R(s_i) > t\},\ n = 0,1,2,... \tag{3}$$

To analyze a replacement process, one should either estimate the probabilities in (3), or equivalently find an estimate for the expected value of, say, N (t) which is defined as

$$E[N(t)] = \sum_{n=0}^{\infty} nP\{N(t) = n\}. \tag{4}$$

This paper chooses to estimate the expected value of N (t).

APPLICATION OF ANTITHETIC RANDOM VARIATES TO SIMULATION OF A SINGLE COMPONENT REPLACEMENT PROCESS

Computer programs used for simulating replacement processes often strain the computer memory. Using the variance reduction techniques, one can improve the output precision without increasing the sample size, or reduce the sample size while maintaining the same level of precision. By output precision it is meant the variance of the sample mean, Var $[\overline{N}(t)]$, where $\overline{N}(t)$ is defined as

$$\overline{N}(t) = \sum_{i=1}^{k} N_i(t) / k. \tag{5}$$

Using antithetic deviates is advantageous in that it reduces the variance of the sample mean without introducing bias in the point estimate. In a simulation experiment, the expected value of a random variable like X is estimated by taking the random sample X_1, X_2,..., X_k from the distribution F (x). Obviously the point estimate $\bar{X}=\sum_{i=1}^{k} X_i/k$ is unbiased for E [X]. The variance of this estimator is generally defined as

$$\mathrm{Var}(\bar{X}) = [\sum_{i=1}^{k} \mathrm{Var}(X_i) + 2\sum_{i<j}\sum \mathrm{COV}(X_i, X_j)]/k^2. \quad (6)$$

If the observations in the sample are correlated, the covariances in (6) will differ from zero. In fact, if negative correlation is introduced in the sampling process, the covariance term in (6) becomes smaller than $\sum \mathrm{Var}(X_i)/k^2$ while $\bar{X}$ remains an unbiased estimator.

Hammerseley and Handscomb[7] have shown the variance reduction property for the sample $X_1, X_2, \ldots, X_k$ (k even) when X_{2i+1} is generated from the random number U_i and X_{2i+2} is generated from $U'_j = 1 - U_j$ where i = 0,1,..., k / 2 -1 and j = i + 1. Under such circumstances, the pair (X_{2i+1}, X_{2i+2}) is called a pair of complementary antithetic random variables.

Based on Mitchell's [8] theorem, George [9] proves that when inverse transformation method is used to generate the random deviates from the distribution of the residual lives, the variance reduction is surely possible in simulation of a replacement process.

The inverse transformation method is not applicable to a number of important probability density functions (pdf). Random deviates from, say, Gamma pdf are commonly generated by the rejection method and as such it will be shown that it is still possible to reduce the variance of the sample mean number of replacements through complementary antithetic random deviates.

VARIANCE REDUCTION IN SIMULATION OF A SINGLE COMPONENT REPLACEMENT PROCESS WHEN REJECTION METHOD IS USED

Let X have density function f_x. To generate random deviates from f_x by the rejection method, an "enveloping function" should be selected. This enveloping function is basically a density function like l_Y (See Figure 1) which is multiplied by a suitable constant like a where $a > 1$. In case such a constant exists, it should be the smallest number larger than 1 which makes al_Y tangent to f_x (See Figure 2). In search for l_Y, it is intended to select l_Y in a way that generating random deviates from it is rather easy.

To generate a random deviate from f_x, a point, P (x, y), is generated randomly under the curve al_Y (See Figure 2). The x - coordinate of point P is a random deviate generated from l_Y and the y - coordinate is a random deviate uniformly distributed between zero

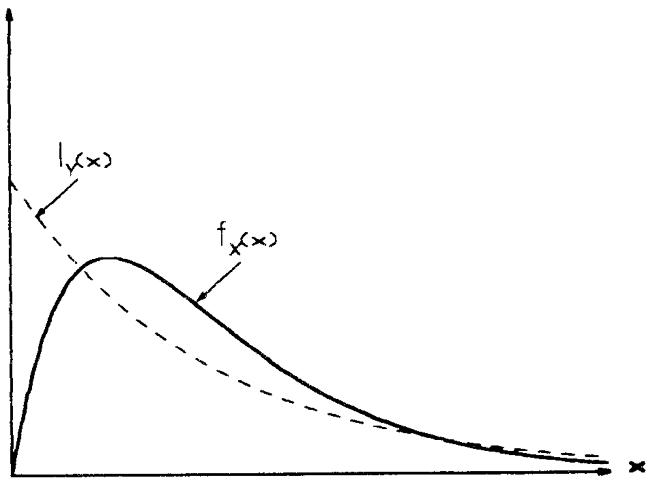

Figure 1. Density functions f_x (x) and l_Y (x).

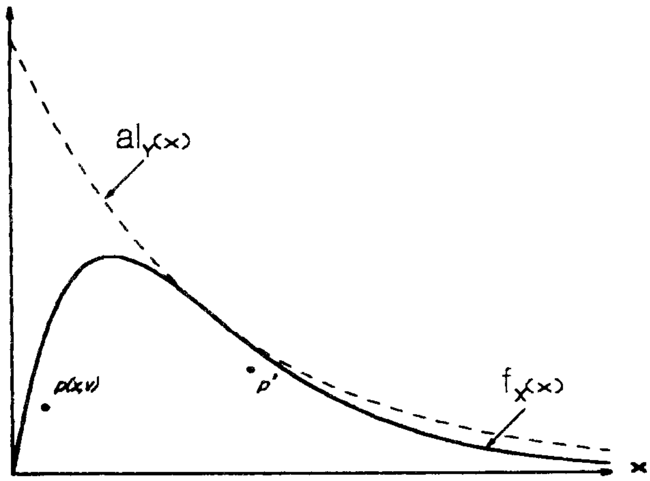

Figure 2. Gamma density function and the enveloping function al_Y (x).

and al_Y (x). In case, the y - coordinate, v, does not exceed f_X (x), the point P is regarded as a success in the sense that x is taken as a random deviate from f_X (See Morgan [10]).

Throughout this paper it is assumed that the inverse transformation method is applicable toward generating deviates from l_Y, as such, to generate x_i requires generating just one random number.

Generating the antithetic random deviate for a successful x, amounts to generating a successful pair of points, say, P and P'. A successful pair of points like P (x, v) and P' (x', v') consists of two points that lie in the region bounded by the two axes and the graph of f_x (See Figure 2) with their x - coordinates being antithetic in the sense that x and x' are generated by using the random numbers U_i and $U'_i = 1 - U_i$ respectively. It is important to note that even though the y - coordinates of such pair of points will be generated by the same random number, they are different because the antithetic random number that makes the x- coordinate of P', also participates in making the y-coordinate.

Theorem. Simultaneous application of the rejection method and complementary antithetic deviates in simulation of a single component replacement process will reduce the variance of the sample mean number of replacements during the time interval [0, t], provided that there exists, at least, one pair of successful points P and P'.

Proof**.** First, it must be shown that

$$COV\,[N_1\,(t, i), N_2\,(t, i)] \le 0, \forall i \tag{7}$$

holds and from (7) one would have

$$\lim_{i\to\infty} COV\,[N_1\,(t, i), N_2\,(t,i)] = COV\,[N_1\,(t), N_2\,(t)\,] \le 0 \tag{8}$$

where, i = 1,2,... represents the number of components available for use in [0, t] including the initial component.

In case there are at least one pair of successful antithetic points, we will show that if N_1 is a nonincreasing function of the random number U_j, then N_2 will be a non- decreasing function of the same random number and vice versa. In fact, if U_j is large, $L^{-1}(U_j)$, i.e., the generated value for the component's life will be large (L is the CDF corresponding to density l_Y). Generating a large value for the component's life implies that N_1 (t, i) is rather small. On the other hand, when U_j is large, $1-U_j$ and from there $L^{-1}(1 - U_j)$ will be small and this means a large value for N_2 (t, i).

Thus, under the condition stated above, we have

$$COV\,[N_1\,(t, i), N_2\,(t, i)] \le 0$$

Since, for a finite time interval, [0, t], the value of $N_1(t, i)$ and N_2 (t, i) are finite for all i, we can write

$$\begin{aligned} &\lim_{i\to\infty} COV\,[N_1\,(t, i), N_2\,(t, i)] = \\ &COV\,[\lim_{i\to\infty} N_1\,(t, i), \lim_{i\to\infty} N_2\,(t, i)] = \\ &COV\,[N_1\,(t), N_2\,(t)] \le 0 \end{aligned} \tag{9}$$

In fact, since each of the covariance terms is non-positive, the limit also is non - positive.

Let i designate the least positive integer for which complementary antithetic random numbers U_i and $1 - U_i$ lead to a successful pair of points P (x, v) and P' (x', v'). When $U_i = u$, the covariance between N_1 (t) and N_2 (t) can be written as

$$COV\,[\,N_1\,(t), N_2\,(t)] = \int_0^1 E\,[[N_1\,(t) - EN_1\,(t)]\,[N_2\,(t) - EN_2\,(t)\,]\,|\,U_i = u]\,du$$

where EN_i (t) is an abreviation of E $[N_i\,(t)]$, i = 1,2. Since, $EN_1\,(t) = EN_2\,(t)$ holds, the last equation can be written as

$$COV[N_1\,(t), N_2\,(t)] = \int_0^1 E\,[[N_1(t) - EN_1(t)]|\,U_i = u]\,du \tag{10}$$

Now, let U_i be a value quite close to 1 denoted by u^* such that $\sum_j X_j > t$. If $U_i < u^*$ holds, $N_1(t)$ will be equal to $i - 1$ and the covariance equation will become

$$\begin{aligned} COV[N_1(t), N_2(t)] &= \int_{u^*}^{1} [[(i-1) - EN_1(t)]\, E[N_2(t) - EN_1(t)] \mid U_i = u]\, du + \int_0^{u^*} E[[N_1(t) - EN_1(t)][N_2(t) - EN_1(t)] \mid U_i = u]\, du \\ &= \int_{u^*}^{1} [[(i-1) - EN_1(t)][EN_2(t) - EN_1(t)] \mid U_i = u]\, du + \int_0^{u^*} E[[N_1(t) - EN_1(t)][N_2(t) - EN_1(t)] \mid U_i = u]\, du \end{aligned} \tag{11}$$

In the first term on the right hand side, we have $EN_2(t) > EN_1(t)$ since the condition $U_i > u^*$ is forced, and at the same time $(i - 1)$ is smaller than $EN_1(t)$. All this means that the first integral leads to a negative value. Forcing the condition $U_i < u^*$ in the second term on the right leads to a non - positive value since $N_1(t)$ becomes larger than usual while $N_2(t)$ becomes smaller than usual. Thus, using the complementary antithetic random numbers leads to a smaller variance for the mean number of replacements during the interval $[0, t]$.

Corollary. Simultaneous implementation of the rejection method and the complementary antithetic random numbers in simulation of a single component replacement process reduces the variance of the mean number of replacements during the time interval $[0, t]$ where each spare and the initial component have accumulated some specified amount of wear, provided, at least, one pair of successful antithetic points is generated.

Proof. Proof is similar to what was shown above, except for the fact that the enveloping density function $l_Y(X)$ now becomes a truncated density like $l_Y(x)$, $x > s > 0$, where s stands for the amount of wear at installation. This proof is valid when for component i the accumulated amount of wear becomes s_i. In fact, the sequence $\{X\}$ must be generated in such a way that none of the X's is smaller than its corresponding s_i which is straightforward.

The algorithm

***Step* 1.** Choose the enveloping density function l_Y in such a way that its random deviates can be generated by inverse transformation method.

***Step* 2.** Use the rejection method to generate a sequence of "successful points" $P_1(x_1, v_1)$, $P_2(x_2, v_2)$, ... , $P_{N_1(t)+1}(x_{N_1(t)+1}, v_{N_1(t)+1})$ such that the sum $X_1 + X_2 + ... + X_{N_1(t)+1}$ is the smallest value equal to or larger than t.

***Step* 3.** Assuming that the sequence of random numbers $U_1, U_2, ..., U_n$, $(n > N_1(t) + 1)$ has led to the sequence of component lives $X_1, X_2, ... X_{N_1(t)+1}$, define the sequence of random numbers $U'_i = 1 - U_i$ where $i = 1,2,..., n$ and use as many of them as needed for generating the random component lives $X'_1, X'_2, ..., X'_{N_2(t)+1}$ such that the sum $X'_1 + X'_2 + ... + X'_{N_2(t)+1}$ is the smallest value equal to or larger than t. If n is not large enough for this condition to hold, then extra random numbers should be generated independently and used.

Following the obove - mentioned steps will result in the pair $(N_1(t), N_2(t))$. This pair would be a complementary antithetic pair provided for some random number like U_i, and its complementary antithetic random number, U'_i, two successful points like $P(x_i, v)$ and $P'(x'_i, v')$ are generated.

COMPUTATIONAL RESULTS

This section is devoted to the presentation of the results from several replacement process simulations with complementary antithetic variates used in conjunction with the acceptance / rejection method.

The time to failure distribution of the components is assumed to be gamma with the scale parameter (basically) $\beta = 1$.

For the shape parameter, α, values in the range 1.1 to 3.5 are considered. Run lengths, t, of 5, 10, and 15 hours are used.

The design of each simulation run is such that in each iteration three values $N_1(t)$, $N_2(t)$, and $N_3(t)$ are generated where,

$N_1(t)$= the number of replacements through time t when the random numbers are generated independently (Column" Base" of Table I),

$N_2(t)$ = the number of replacements through time t when times to failures are generated randomly and independently of the "Base" (Column "Independent" of Table I),

$N_3(t)$ = the number of replacements through time t when the sequence of random numbers used in the "Base" are used for complementary antithetic sampling (Column "Antithetic" of Table I). The number of runs for determination of each $N_i(t)$ is set equal to 10,000.

As intended, the values from Base and Independent runs should not be correlated while the values from the Base and Antithetic runs should be negatively correlated. The values from the Independent runs are generated and combined with the Base runs to provide a comparison with the correlated runs Base and Antithetic. The sample means and variances of the number of replacements are shown in Table I. The sample coefficients of correlation and the sample variances between Base and Antithetic and between Base and Independent are tabulated in the Antithetic and Independent columns.

The sample coefficient of correlation between Base and Independent runs was expectedly near zero, while it ranged from -0.04 to -0.47 between Base and Antithetic runs. In fact, the value of the parameter α has a decisive effect on the magnitude of the induced negative correlation and hence the reduction in the variance of the sample mean (VSM). Such effect is reciprocal to the magnitude of the shape parameter.

Table I contains a column that (for an ordinary renewal sequence) shows the value of lim E [N (t)] as t goes to infinity [11]. As can be seen for each of the simulations shown in Table I, the difference between the $\lim_{t\to\infty} E[N(t)] = t/\mu + (\sigma^2 - \mu^2)/(2\mu^2)$ and the

TABLE I. Simulation of Replacement Processes

Input			Mean			Lim	Variance			Corr. Coeff.		VSM *1.0E-4		Var.
α	β	t	Base	Ind.	Anti.	E[N(t)]	Base	Ind.	Anti.	Ind.	Anti.	Ind.	Anti	Red.
1.10	1.00	10	9.03	9.03	9.07	9.08	8.07	8.12	8.36	-0.01	-0.47	4.00	2.20	%45
1.50	1.00	10	6.50	6.48	6.50	6.50	4.52	4.64	4.42	-0.01	-0.30	2.27	1.57	%31
2.00	1.00	10	4.75	4.74	4.73	4.75	2.62	2.57	2.58	0.01	-0.21	1.31	1.02	%22
2.50	1.00	10	3.69	3.70	3.70	3.70	1.68	1.68	1.68	0.01	-0.15	0.85	0.72	%15
3.00	1.00	10	3.00	3.00	3.00	3.00	1.17	1.20	1.19	0.02	-0.07	0.60	0.55	%8.3
3.50	1.00	10	2.51	2.49	2.50	2.50	0.88	0.91	0.48	-0.01	-0.04	0.44	0.42	%4.5
2.00	1.00	5	2.26	2.25	2.23	2.25	1.31	1.32	1.31	-0.01	-0.21	0.65	0.52	%20
2.00	1.00	15	7.25	7.27	7.25	7.25	3.83	3.84	3.84	-0.01	-0.20	1.91	1.53	%19.9
2.50	1.50	10	2.34	2.36	2.36	2.37	1.12	1.14	1.15	-0.02	-0.14	0.55	0.49	%10.9
2.50	2.00	10	1.69	1.69	1.70	1.70	0.84	0.86	0.86	0.02	-0.15	0.43	0.37	%13.9
2.50	2.50	10	1.29	1.30	1.30	1.30	0.71	0.72	0.72	0.02	-0.17	0.37	0.30	%18.9

estimated means of N(t) is negligible (μ and σ^2 are the mean and variance of the component life's distribution function).

In order to show that how well the computational results match the theoretical results, the antithetic estimated mean number of renewals was compared with the expected value of N(t). In fact, the expected value of N(t), where possible, was calculated through the following relation [12].

$$E[N(t)] = \frac{\beta t}{\alpha} + \frac{1}{\alpha}\sum_{r=1}^{\alpha-1} \frac{\varepsilon^r}{1-\varepsilon^r}\left(1 - e^{-\beta t(1-\varepsilon^r)}\right),\ \alpha = 1,2,...$$

in which ε^r is defined as $\varepsilon^r = e^{2\pi i r/\alpha}$. Table II shows the results of such comparison for the case of an integer α.

TABLE II. A Comparison of the Computational Results Against the Theoretical Results

α	t	Estimated Mean (Ant.)	Exact value of E [N (t)]	Difference (%)
2	5	2.23	2.25	0.8
2	10	4.73	4.75	0.4
2	15	7.25	7.25	0.0
3	10	3.00	3.07	2.2

CONCLUSIONS AND FUTURE EXTENSIONS

This paper has proposed a pragmatic approach to the problem of generating antithetic random deviates in conjunction with the acceptance-rejection method. This approach was experimentally tested for the case of a single component replacement process for which the possibility of reducing the variance of the mean number of replacements was also established analytically.

Throughout the test runs, Gamma density function (with $\alpha > 1$) was chosen to represent the time to failure of the component. According to the results, the magnitude of variance reduction is highly dependent on the value of the shape parameter, α, of Gamma. In fact, the flatter the shape of the pdf, the higher is the magnitude of variance reduction.

The algorithm developed in this work can be modified and applied to other types of antithetic random numbers to shed some more light on the possibility of enhancing the magnitude of variance reduction for large values of the shape parameter of a Gamma density function.

To enhance the magnitude of variance reduction, different schemes of applying the rejection method toward generating the random deviates (in particular, the Cheng's [10] method for gamma pdf) can be examined. In fact, different schemes adopt different "enveloping functions" which can drastically reduce the probability of rejection and hence increase the magnitude of variance reduction.

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