FUNDAMENTAL SOLUTIONS OF DYNAMIC POROELASTICITY AND GENERALIZED THERMOELASTICITY

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Abstract Fundamental solutions of dynamic poroelasticity and generalized thermoelasticity are derived in the Laplace transform domain. For poroelasticity, these solutions define the solid displacement field and the fluid pressure in fluid-saturated media due to a point force in the solid and an injection of fluid in the pores. In addition, approximate fundamental solutions for short times are derived by analytically inverting the Laplace transform expressions. Finally numerical results are presented to highlight the essential features of the problem as well as to investigate the accuracy of the time domain solutions.

چکیده در این مقاله حل اساسی در مسائل دینامیک محیط های متخلخل اشباع و ترموالاستیسیته در حوزهٔ تبدیل فوریه بدست می‌آیند. در مسائل محیط‌های متخلخل این حل معرف میدان تغییر مکان در فاز جامد و میدان فشار در فاز مایع در اثر یک نیروی متمرکز در اسکلت جامد و تزریق مایع بداخل حفره هاست. این حل برای فرموله کردن یک مدل عددی المانهای فرضی نقش اصلی دارند. علاوه براین با معکوس کردن تحلیلی روابط بدست آمده حل اساسی فوق بصورت بسته برای زمانهای کوچک در حوزه زمان بدست می آیند. برای نشان دادن مشخصات این حل اساسی و بخصوص موضوع اثر متقابل فازهای جامد و مایع تعدادی نتیجه عددی براساس روابط تحلیلی فوق ارائه میشوند. همچنین دقت روابط تقریبی برای حل اساسی بامقایسه نتایج حاصل از معکوس کردن عددی حل حوزه فرکانس وحل بسته نشان داده میشوند.

INTRODUCTION

Dynamic poroelasticity has applications in numerous branches of science and engineering, such as geophysics, biomechanics, civil and mechanical engineering. The three-dimensional theory of this problem was first developed by Biot [1, 2]. Biot postulated a potential energy for poroelastic media and utilized Lagrange's equations to derive a set of coupled differential equations governing the motions of the solid and fluid phases. According to this theory, a dynamic disturbance in a porous medium generates one transverse (shear) and two longitudinal (pressure) waves. The first longitudinal wave, sometimes denoted as $P1$-wave, is characterized by lightly attentuated solid and fluid motions that are in phase. The second longitudinal wave ($P2$-wave), on the other hand, is associated with highly-attenuated and out-of-phase motions of the constituents. The existence of these waves has also been demonstrated experimentally [3]. More modern theories of continuum mechanics, such as the theory of mixtures [4, 5] have resulted in equations with essentially similar characteristics [6, 7, 8, 9, 10].

The first attempt to obtain the fundamental solutions pertaining to Biot's equation was made by Burridge and Vargas [11] who, in addition to presenting a general solution procedure similar to that of Deresiewicz [12] used the saddle point method to obtain displacements at large distances due to a point force in the solid. Later, Norris [13] derived steady-state fundamental solutions for a point force in the solid and a point force in the fluid. He also obtained explicit asymptotic approximations for far-field displacements, as well as those for high and low frequency responses. More recently, Kaynia and Banerjee [14] used a solution scheme similar to that of Norris [13] and derived explicit expressions for the fundamental solutions in the Laplace transform domain as well as transient short-time solutions.

Biot's equations are in terms of two displacement fields, namely those of solid and fluid; therefore, they are somewhat inconvenient for the solution of practical boundary value problems. For this reason, Biot's equations are sometimes recast in terms of the solid displacement field and the fluid pressure (u - p model). This can, however, be achieved only in a transformed domain with the resulting equations being dependent on the transformed parameter. Another advantage of this type of formulation is that the resulting coupled equations resemble those of thermoelasticity for which the fundamental solutions are available [15, 16, 17]. This form of poroelasticity equations has been used by Bonnet [18] and by Boutin et al. [19] to derive steady-state fundamental solutions of poroelasticity by the Kupradze method. However, Bonnet develops a poroelastic formulation which is identical to that of classical thermoelasticity, thus predicting infinite propagational velocities. On the other hand, the work of Boutin et al. is based upon a set of governing equations which fail to reduce to the well - established consolidation theory [20] in the absence of inertial effects.

In this paper, while a similar approach is adopted, both of the above-mentioned difficulties are overcome. Explicit expressions are derived in the Laplace transform domain for the fundamental solutions of dynamic poroelasiticy. These solutions define the solid displacement field and the fluid pressure due to a suddenly applied point force in the solid and a sudden injection of fluid into the pores. The expressions for the fundamental solutions are then inverted analytically to obtain approximate transient short-time solutions. Finally, a number of results are presented to demonstrate the charactersitics of the waves and the interaction between the two phenomena present in the behavior of porous media, namely the wave propagation and the diffusions.

The expressions derived in this paper as well as the observations made on the dynamic behavior of porous media are equally applicable to generalized thermoelasticity problems. In the theory of generalized thermoelasiticy [21, 22] the paradox of an infinite velocity for the propagation of a disturbance, which is associated with the classical theory of dynamic thermoelasticity, is eliminated. The analogy between the equations of this theory and those of dynamic poroelasticity is also addressed in the present paper.

BASIC EQUATIONS

Following the formulations outlined by Zienkiewicz et al. [23] and by Boutin et al. [19], one can write the equations expressing respectively, the consititutive equation of a poroelastic solid, the generalized Darcy's law, the conservation of momentum and the continuity equation, as

$$\sigma_{ij} = \lambda u_{l,l}\delta_{ij} = \mu(u_{i,j} + u_{j,i}) - \beta p \delta_{ij} \qquad (1)$$

$$p_{,i} = - b\dot{w}_i - \rho_f \ddot{u}_i - m\ddot{w}_i \qquad (2)$$

$$\sigma_{ij,j} + f_i = \rho\ddot{u}_i + \rho_f \ddot{w}_i \qquad (3)$$

$$\dot{w}_{l,l} = - \beta\dot{u}_{l,l} - \frac{1}{M_p}\dot{p} + \gamma \qquad (4)$$

where p represents the pore pressure, u_i denotes the displacement of the solid skeleton and w_i denotes the average displacement of the fluid relative to the solid. Furthermore, λ and μ are the drained Lame′ constants, ρ is that of the solid-fluid mixture and m is another mass parameter which has been shown to be equal to ρ_f/n [23] with n denoting porosity. The material parameters β and M_p describe relative compressibilities of the constituents and are given by

$$\beta = 1 - \frac{K_d}{K_s} \qquad (5)$$

$$\frac{1}{M_p} = \frac{n}{K_f} + \frac{\beta - n}{K_s} \qquad (6)$$

where K_f and K_s are the bulk moduli of the fluid and the solid grain and K_d is that of the solid skeleton (drained bulk modulus). Finally $b = \eta / \kappa$ is the resistivity coefficient of the medium, with η and κ denoting the fluid viscosity and the permeability of the solid skeleton, and f_i and γ represent the body force and the rate of fluid injection into the medium, respectively. The indices i, j, and l vary from one to three for three-dimensional domains.

Equations (1) to (4), in the absence of f_i and γ, are equivalent to Biot's equations of dynamic poroelasticity [2]. This can simply be verified by eliminating p and σ_{ij} from these equations to arrive at two coupled differential equations in terms of the solid and fluid displacements. Eliminating w_i, on the other hand, is not possible except in a transformed space.

Taking the Laplace transform of Equations (1) to (4) and eliminating w_i, one obtains new equations which can be expressed as

$$(\lambda + \mu)\,\bar{u}_{j,ij} + \mu\bar{u}_{i,jj} - \beta_p\bar{p}_{,i} - \rho_p s^2\bar{u}_i + \bar{f}_i = 0 \qquad (7)$$

$$\theta_p\bar{p}_{,jj} - \frac{s}{M_p}\bar{p} - \beta_p s\bar{u}_{j,j} + \bar{\gamma} = 0 \qquad (8)$$

where the overbar denotes the Laplace transformation. $\theta_p = 1/(b + ms)$, $\beta_p = \beta - \rho_f s\theta_p$, $\rho_p = \rho - \rho_f^2 s\theta_p$ and s is the Laplace transform parameter. Under quasistatic conditions, these equations reduce directly to those associated with three-dimensional consolidation theory.

Additionally, the new form of equations in the Laplace transform space (Equations 7 and 8) are especially helpful in establishing the analogy between dynamic poroelasticity and generalized thermoelasticity. According to Lord and Shulman [21] differential equations of the latter are expressed as

$$(\lambda+\mu)u_{j,ij}+\mu u_{i,jj}-(3\lambda+2\mu)\alpha_T T_{,i}-\rho\ddot{u}_i+f_i=0 \tag{9}$$

$$kT_{,jj}-\rho c_\varepsilon(\dot{T}-\tau_o\ddot{T})+(3\lambda+2\mu)\alpha_T T_o(\dot{u}_{jj}+\tau_o\ddot{u}_{j,j})+\gamma+\tau_o\dot{\gamma}=0 \tag{10}$$

where T denotes the absolute temperature and T_o, c_ε, α_T and k denote the stress-free temperature of the body, the specific heat at constant strain, the coefficient of thermal expansion and the thermal conductivity, respectively. τ_o is the relaxation time which represents the time lag needed to establish steady-state heat conduction in an element of volume when a temperature gradient is suddenly imposed on that element. Finally, λ, μ are the isotropic Lame′ parameters, while ρ is the mass density of the thermoelastic material. Body forces and heat sources are represented by f_i and γ, respectively.

Taking the Laplace transform of Equations (9) and (10) one obtains

$$(\lambda+\mu)\bar{u}_{j,ij}+\mu\bar{u}_{i,jj}-\beta\bar{T}_{,i}-\rho s^2\bar{u}_i+\bar{f}_i=0 \tag{11}$$

$$\theta_T\bar{T}_{,jj}-\frac{s}{M_T}\bar{T}-\beta T_o s\bar{u}_{j,j}+\bar{\gamma}=0 \tag{12}$$

where $\beta=(3\lambda+2\mu)\alpha_T$, $\theta_T=k/(1+\tau_o s)$ and $M_T=1/\rho c_\varepsilon$.

Equations (7) and (8) of poroelasticity and Equations (11) and (12) of generalized thermoelasticity are quite similar, even on the type of dependency of the coefficients on s. It may be argued, however, that whereas β_p and ρ_p in Equations (7) and (8) are dependent on s, their counterparts in Equations (11) and (12), i.e., β and ρ, are constants. This minor difference can simply be explained by noticing that in a poroelastic material ρ_f is nonzero (unlike in thermoelasticity); if ρ_f is set to zero then β_p and ρ_p reduce, respectively, to β and ρ which are constants as in thermoelasticity. In other words, Equations (7) and (8) of poroelasticity equally apply to generalized thermoelasticity with ρ_f equal to zero in the latter case.

TRANSFORMED FUNDAMENTAL SOLUTIONS

The objective of this section is to obtain the fundamental solutions associated with Equations (7) and (8). These solutions define the solid displacement field and the fluid pressure due to a unit point force in the solid and a unit rate of fluid injection into the medium. Similar definitions apply to generalized thermoelasticity where the quantities of interest are, of course, the displacements and temperature.

For a continuous unit point force in the jth direction suddenly applied at the origin, the body force $f_i(X,t)$ can be expressed as $\delta(X)H(t)$, where H(t) is the Heaviside step function; then its Laplace transform, is $s\delta(X)$ Similarly for a unit rate of fluid injection at the origin, one has $\gamma(X,t)=\delta(X)H(t)$, the Laplace transform of which is $\bar{\gamma}(X,s)=s\,\delta(X)$. Now following the Kupradze method [16] one can write Equations (7) and (8) as

$$\underset{\sim}{B}\,\underset{\sim}{\bar{G}}+s\underset{\sim}{I}\delta(X)=0 \tag{13}$$

where $\underset{\sim}{B}$, $\underset{\sim}{\bar{G}}$ and $\underset{\sim}{I}$ are 4 x 4 matrices representing the differential operator, the transformed fundamental solution and the unit matrix, respectively. In particular, the elements of $\underset{\sim}{B}$ are

$$B_{ij}=(\lambda+\mu)\frac{\partial^2}{\partial x_i\partial x_j}+\delta_{ij}(\mu\Delta-\rho_p s^2) \tag{14a}$$

$$B_{i4}=-\beta_p\frac{\partial}{\partial x_i} \tag{14b}$$

$$B_{4j}=-\beta_p s\frac{\partial}{\partial x_j} \tag{14c}$$

$$B_{44}=\theta_p\Delta-\frac{s}{M_p} \tag{14d}$$

where Δ is the Laplacian operator. The determinant of B is given by

$$\det\underset{\sim}{B}=\theta\mu^2(\lambda+2\mu)(\Delta-\lambda_1^2)(\Delta-\lambda_2^2)(\Delta-\lambda_3^2)^2 \tag{15}$$

where

$$\lambda_3^2=\frac{\rho_p}{\mu}s^2 \tag{16}$$

and λ_1^2 and λ_2^2 are obtained from the following equations

$$\lambda_1^2+\lambda_2^2=\kappa_1^2+\frac{s}{\theta_p}\left(\frac{1}{M_p}+\frac{\beta_p^2}{\lambda+2\mu}\right) \tag{17}$$

$$\lambda_1^2\lambda_2^2=\frac{s}{\theta_p}\frac{1}{M_p}\kappa_1^2 \tag{18}$$

and κ_1^2 is given by

$$\kappa_1^2=\frac{\rho_p}{\lambda+2\mu}s^2 \tag{19}$$

If the matrix differential operator $\underset{\sim}{B'}$ denotes the transpose of the cofactor of $\underset{\sim}{B}$ one can show that its elements can be expressed as

$$B'_{ij}=\mu(\lambda+2\mu)\left\{\delta_{ij}\theta_p(\Delta-\lambda_1^2)(\Delta-\lambda_2^2)(\Delta-\lambda_3^2)\right.$$
$$\left.-\frac{1}{\lambda+2\mu}\frac{\partial^2}{\partial x_i\partial x_j}\left[(\lambda+\mu)\left(\theta_p\Delta-\frac{s}{M_p}\right)-\beta_p^2 s\right](\Delta-\lambda_3^2)\right\} \tag{20a}$$

$$B'_{i4}=\beta_p\mu^2\frac{\partial}{\partial x_i}(\Delta-\lambda_3^2)^2 \tag{20b}$$

$$B'_{4j}=\beta_p\mu^2 s\frac{\partial}{\partial x_i}(\Delta-\lambda_3^2)^2 \tag{20c}$$

$$B'_{44}=\mu^2(\lambda+2\mu)(\Delta-\kappa_1^2)(\Delta-\lambda_3^2)^2 \tag{20d}$$

Defining φ as $\overline{G} = s\underset{\sim}{B}'\varphi$ one can write

$$\sum_{k=1}^{4} B_{ik}B'_{kj}\varphi \equiv \delta_{ij}\det\underset{\sim}{B}\varphi = \theta_p\mu^2(\lambda+2\mu)\delta_{ij}(\Delta-\lambda_1^2)(\Delta-\lambda_2^2)(\Delta-\lambda_3^2)^2\varphi = 0$$

Now if ψ is defined as $\psi = \theta_\rho\mu^2(\lambda+2\mu)(\Delta-\lambda_3^2)\varphi$ then

$$(\Delta-\lambda_1^2)(\Delta-\lambda_2^2)(\Delta-\lambda_3^2)\psi = 0 \qquad (21)$$

the solution of which is

$$\psi(r,s) = \frac{1}{4\pi}\sum_{k=1}^{3}\frac{e^{-\lambda_k r}}{(\lambda_{k+1}^2-\lambda_k^2)(\lambda_{k+2}^2-\lambda_k^2)r} \qquad (22)$$

with $\lambda_4 = \lambda_1$, $\lambda_5 = \lambda_2$, and $r^2 = x_i x_i$.

Finally denoting the matrix operator $B'' = \dfrac{\underset{\sim}{B'}}{(\Delta-\lambda_3^2)}$

one obtains

$$\underset{\sim}{\overline{G}} = sB'\varphi = sB''(\Delta-\lambda_3^2)\varphi = \underset{\sim}{B}\frac{s}{\theta_p\mu^2(\lambda+2\mu)}\psi \qquad (23)$$

The elements of $\underset{\sim}{G}$ are obtained by substituting equation (22) by equation (23) and carrying out the necessary operations; the result is

$$\overline{G}_{ij} = \sum_{k=1}^{3}\left\{\left[\delta_{3k}s\frac{\delta_{ij}}{4\pi\mu}+\eta_k s\frac{\partial^2}{\partial x_i\partial x_j}\right]\frac{e^{\lambda_k r}}{r}\right\} \qquad (24a)$$

$$\overline{G}_{i4} = \sum_{k=1}^{3}\left\{\beta_k s\frac{\partial}{\partial x_i}\frac{e^{-\lambda_k r}}{r}\right\} \qquad (24b)$$

$$\overline{G}_{4j} = \sum_{k=1}^{3}\left\{\beta_k s^2\frac{\partial}{\partial x_j}\frac{e^{-\lambda_k r}}{r}\right\} \qquad (24c)$$

$$\overline{G}_{44} = \sum_{k=1}^{3}\left\{\gamma_k s\frac{e^{-\lambda_k r}}{r}\right\} \qquad (24d)$$

where

$$\eta_k \frac{(-1)^k(1-s/M_p\theta_p\lambda_k^2)(\delta_{1k}+\delta_{2k})}{4\pi(\lambda+2\mu)(\lambda_2^2-\lambda_1^2)} \quad \frac{\delta_{3k}}{} \qquad (25)$$

$$\beta_k = \beta_p\frac{(-1)^k(\delta_{1k}+\delta_{2k})}{4\pi\theta_p(\lambda+2\mu)(\lambda_2^2-\lambda_1^2)} \qquad (26)$$

$$\gamma_k = \frac{(-1)^k(\lambda_k^2-k_1^2)(\delta_{1k}+\delta_{2k})}{4\pi\theta_p(\lambda_2^2-\lambda_1^2)} \qquad (27)$$

The elements $\overline{G}_{ij}$ define, in the Laplace transform domain, the solid displacement field due to unit point forces in each of the three orthogonal directions; whereas those elements associated with $\overline{G}_{4j}$ define the corresponding fluid pressure. Also $\overline{G}_{i4}$, in the Laplace transform domain, represent the solid displacement field due to a unit rate of fluid injection and $\overline{G}_{44}$ determines the corresponding fluid pressure. To obtain variations of these functions one needs to evalute their inversion. Due to the complexity of these functions analytical inversion is not usually possible, so recourse has to be made to numerical schemes. Nevertheless, for highly permeable media and for short times the expressions of the transformed fundamental solutions (Equation 24) can be inverted analytically. The details of such calculations are presented in the next section.

TRANSIENT FUNDAMENTAL SOLUTIONS

Following the method proposed by Hetnarski [24] one can invert analytically the Laplace transformed fundamental solutions (Equation 24) to obtain the corresponding approximate expression for short times. To that end, one needs to take $p = \frac{1}{s}$, use a Maclaurin series to expand the parameters involved in Equation (24) in powers of p and finally use the tabulated inversion formulae to obtain the transient solutions. For the sake of clarity, the procedure is outlined here for $\overline{G}_{44}$ (i. e. for the fluid pressue due to a sudden injection of fluid, with unit rate, into the medium).

Using equation (24) one can write

$$\overline{G}_{44}(r,s) = s\gamma_1\frac{e^{-\lambda_1 r}}{r} + s\gamma_2\frac{e^{-\lambda_2 r}}{r} \qquad (28)$$

where γ_1 and γ_2 are given by (Equation 27)

$$\gamma_1 = \frac{1}{4\pi\theta_p}\frac{\lambda_1^2-\kappa_1^2}{\lambda_2^2-\lambda_1^2} \qquad (29)$$

$$\gamma_2 = \frac{1}{4\pi\theta_p}\frac{\lambda_2^2-\kappa_1^2}{\lambda_2^2-\lambda_1^2} \qquad (30)$$

and λ_1^2 and λ_2^2 which are the solution of Equations (17) and (18), are given by

$$\lambda_1^2\lambda_2^2 = \frac{1}{2}\left[\kappa_1^2+\frac{s}{\theta_p}\left(\frac{1}{M_p}+\frac{\beta_p^2}{\lambda-2\mu}\right)\right] \pm\frac{1}{2}\left\{\left[\kappa_1^2+\frac{s}{\theta_p}+\left(\frac{1}{M_p}+\frac{\beta_p^2}{\lambda+2\mu}\right)\right]^2-4\frac{s}{\theta_p}\frac{1}{M_p}\kappa_1^2\right\}^{\frac{1}{2}} \qquad (31)$$

Using the expressions defining θ_p, β_p^2 and k_1^2 one can show that

$$\kappa_1^2+\frac{s}{\theta}\left(\frac{1}{M_p}+\frac{\beta_p^2}{\lambda+2\mu}\right) = \frac{1}{p^2}\left(a_1+\frac{b}{m}a_2 p\right) \qquad (32)$$

$$\left[k_1^2+\frac{s}{\theta_p}\left(\frac{1}{M_p}+\frac{\alpha_1^2}{\lambda+2\mu}\right)\right]^2-4\frac{s}{\theta_p}\frac{1}{M_p}k_1^2=\frac{1}{p^4}\left[b_1+\left(\frac{b}{m}\right)b_2p+\left(\frac{b}{m}\right)^2b_3p^2\right] \tag{33}$$

where a_1, a_2, b_1, b_2 and b_3 are given in the Appendix.

Using relations (32) and (33) and Maclaurin series to expand second term on the right-hand side of equation (31) one obtains

$$\lambda_1^2 \sim \frac{1}{p^2}(d_0+d_1p+d_2p^2+d_3p^3+d_4p^4) \tag{34}$$

$$\lambda_2^2 \sim \frac{1}{p^2}(d_5+d_6p+d_7p^2+d_8p^3+d_9p^4) \tag{35}$$

where the expressions for d_0 through d_9 are given in the Appendix.

A further application of Maclaurin series to obtain the expansion for the square root of relations (34) and (35) results in

$$\lambda_1 \sim \frac{f_1}{p}+f_2+f_3p+f_4p^2+f_5p^3$$

$$\lambda_2 \sim \frac{g_1}{p}+g_2+g_3p+g_4p^2+g_5p^3$$

where the f and g coefficients are given in the Appendix.

A final application of Maclurin series to obtain the expansion for the inverse of $(\lambda_2^2-\lambda_1^2)$ followed by the algebraic multiplications involved in equations (29) and (30) leads to an expression for $\overline{G}_{44}$ (equation 28) as

$$\overline{G}_{44}=\frac{m}{4\pi r}\left[q_0+\frac{q_1}{s}+\frac{q_2}{s^2}+\frac{q_3}{s^3}+\frac{q_4}{s^4}\right]e^{-f_2r}e^{-f_1rs}e^{-f_3\frac{r}{s}}-\left(q_5+\frac{q_6}{s}+\frac{q_7}{s^2}+\frac{q_8}{s^3}+\frac{q_9}{s^4}\right)e^{-g_2r}e^{-g_1rs}e^{-g_3\frac{r}{s}}\Big] \tag{38a}$$

where q_0 to q_9 (as well as f_1 to f_3 and g_1 to g_3) are constant parameters which depend only on the material properties. Explicit expressions for these parameters are given in the Appendix.

Utilizing similar procedures, one can obtain the following expressions for the other components of the Green's tensor

$$\overline{G}_{i4}=\frac{1}{4\pi(\lambda+2\mu)}\frac{x_i}{r^2}\left[-\left(\frac{A_1}{s}+\frac{A_2}{s^2}+\frac{A_3}{s^3}+\frac{A_4}{s^4}\right)e^{-f_1rs}e^{-f_2r}e^{-f_3\frac{r}{s}}+\left(\frac{B_1}{s}+\frac{B_2}{s^2}+\frac{B_3}{s^3}+\frac{B_4}{s^4}\right)e^{-g_1rs}e^{-g_2r}e^{-g_3\frac{r}{s}}\right] \tag{38b}$$

$$\overline{G}_{4j}=\frac{1}{4\pi(\lambda+2\mu)}\frac{x_j}{r^2}\left[-\left(A_1+\frac{A_2}{s}+\frac{A_3}{s^2}+\frac{A_4}{s^3}+\frac{A_5}{s^4}\right)e^{-f_1rs}e^{-f_2r}e^{-f_3\frac{r}{s}}+\left(B_1+\frac{B_2}{s}+\frac{B_3}{s^2}+\frac{B_4}{s^3}+\frac{B_5}{s^4}\right)e^{-g_1rs}e^{-g_2r}e^{-g_3\frac{r}{s}}\right] \tag{38c}$$

$$\overline{G}_{ij}=\frac{1}{4\pi(\lambda+2\mu)}\left[\left(\frac{C_{ij}^1}{s}+\frac{C_{ij}^2}{s^2}+\frac{C_{ij}^3}{s^3}+\frac{C_{ij}^4}{s^4}\right)e^{-f_1rs}e^{-f_2r}e^{-f_3\frac{r}{s}}\right.$$
$$\left.\left(\frac{C_{ij}^5}{s}+\frac{C_{ij}^6}{s^2}+\frac{C_{ij}^7}{s^3}+\frac{C_{ij}^8}{s^4}\right)e \qquad \frac{r}{s}\right]$$
$$-\frac{1}{4\pi\mu}\left[\frac{C_{ij}^{11}-\frac{1}{r}\delta_{ij}}{s}+\frac{C_{ij}^{12}}{s^2}+\frac{C_{ij}^{13}}{s^3}+\frac{C_{ij}^{14}}{s^4}\right]e^{-h_1rs}e^{-h_2r}e^{-h_3\frac{r}{s}} \tag{38d}$$

where the A, B, C and h coefficients are given in the Appendix.

Equation (38) can now easily be inverted by the use of the following formulae [25]:

a) $f_3<0$:

$$L^{-1}\left\{e^{-f_2r}e^{-f_1rs}e^{-f_3r/s}\right\}=e^{-f_2r}\left[\delta(t^*)+\sqrt{\frac{a}{t^*}}I_1\left(2\sqrt{at^*}\right)\right]H(t^*)$$
$$L^{-1}\left\{s^{-n}e^{-f_2r}e^{-f_1rs}e^{-f_3r/s}\right\}=e^{-f_2r}\left[\left(\frac{t^*}{a}\right)^{\frac{n-1}{2}}I_{n-1}\left(2\sqrt{at^*}\right)\right]H(t^*) \tag{39}$$

b) $f_3>0$:

$$L^{-1}\left\{e^{-f_2r}e^{-f_1rs}e^{-f_3r/s}\right\}=e^{-f_2r}\left[\delta(t^*)-\sqrt{\frac{a}{t^*}}J_1\left(2\sqrt{at^*}\right)\right]H(t^*)$$
$$L^{-1}\left\{s^{-n}e^{-f_2r}e^{-f_1rs}e^{-f_3r/s}\right\}=e^{-f_2r}\left[\left(\frac{t^*}{a}\right)^{\frac{n-1}{2}}J_{n-1}\left(2\sqrt{at^*}\right)\right]H(t^*) \tag{40}$$

where $t^*=t-f_1r$, $a=|f_3|r$ and r is the euclidean distance from the source. Also, $H(t)$, J and I denote the Heaviside step function, the Bessel function of the first kind and the modified Bessel function of the first kind, respectively. The result of these operations is then the approximate transient fundamental solutions.

Equations (38), defining the fundamental solutions of dynamic poroelasticity (as well as generalized thermoelasticity), demonstrate the existence of three damped waves: two longitudinal waves with velocities $1/f_1$ and $1/g_1$ and one transverse wave with velocity $1/h_1$ However, while the longitudinal waves are present in all components of $\underset{\sim}{\overline{G}}$, only $\underset{\sim}{\overline{G}}_{ij}$ contains the transverse response. The damping effects in these waves are reflected through the dissipation factors f_2, g_2 and h_2. Numerical results show that, in general, g_2 is much larger than f_2 and h_2. This implies that the second longitudinal wave is highly dissipative to the extent that it can hardly propagate beyond a close neighborhood of the source. In porous media with practically infinite permeability, dissipation factors vanish and the three waves travel without any dissipation effects.

An interesting feature displayed by the transient fundamental solutions is the presence of pulses in those components defining the pressure, i.e., Equations (38a) and (38c). These pulses, in the form of Dirac delta functions, are associated with the arrival of the two dilatational waves.

The interaction between the two phenomena of wave propagation and diffusion in dynamic poroelasticity (as well as in generalized thermoelasticity) is another interesting subject. As numerical results in the next section show, the second dilatational wave is immediately followed by the diffusion. This observation is particularly valuable in thermoelasticity as it removes an irrational consequence of the classical theory of thermoelasticity which, by eliminating the second dilatational wave, implies an infinite velocity for the propagation of a disturbance.

NUMERICAL RESULTS

A number of results are presented in this section to highlight certain characteristics of the dynamic behavior of fluid-saturated porous media. In addition, the accuracy of the analytical solutions is investigated by a comparison with numerical inversion of Laplace transform solutions.

The material properties are selected such that the numerical results clearly show the two phenomena of wave propagation and diffusion and their interaction. These propeties are

$$\lambda = 0.274 \times 10^4 \mathrm{N/m^2}, \mu = 0.585 \times 10^4 \mathrm{N/m^2}$$

$$M = 0.997 \times 10^4 \mathrm{N/m^2}\ ,\ \alpha = 0.83\ ,\ f = 0..195$$

$$\rho_f = 1000\, Kg/m^3\ ,\ \rho = 2273\, kg/m^3\ ,\ m = 5128\, Kg/m^3$$

$$b = 10^5\, \mathrm{Ns/m^4}$$

Except for the order of magnitude of λ, μ, M and b, these properties are those measured by [26] which were converted by [11] to match the parameters appearing in Equations (1) to (4).

Figures 1 to 4 show four components of the fundamental solutions in the medium, calculated by numerically inverting the Laplace transform expressions (Equations 24). To obtain Figures 1 and 2, a $1N$ point force in the x-direction was applied in the solid at (0,0,0) and the horizontal displacement, G_{11}, and the fluid pressure, G_{41}, were evaluated at the point (0.5,0,0)m. To obtain Figures 3 and 4, on the other hand, the fluid was injected at a rate of $1 m^3/s$

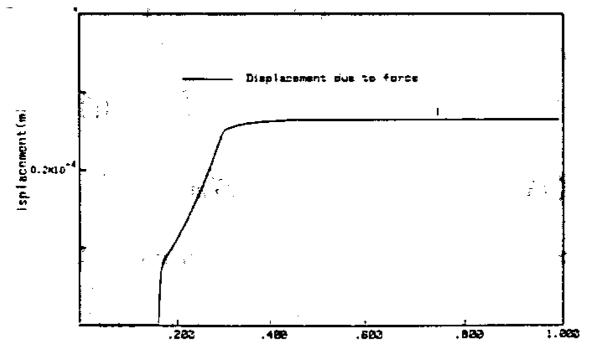

Figure 1. G_{11} : Horizontal solid displacement due to a unit horizontal force.

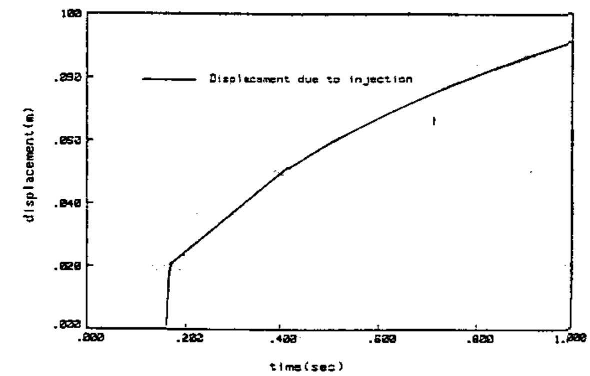

Figure 3. G_{14}: Horizontal solid displacement due to a unit rate of fluid injection.2

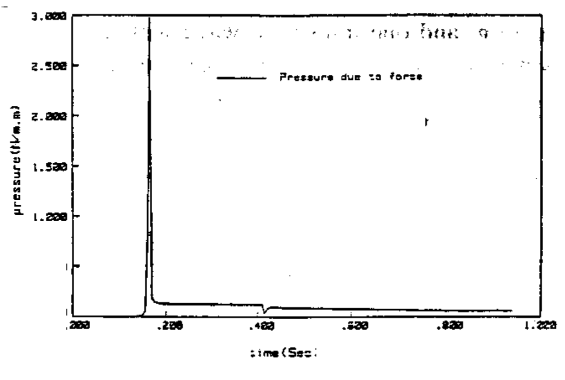

Figure 2. G_{41} : Fluid pressure due to a unit horizontal force.

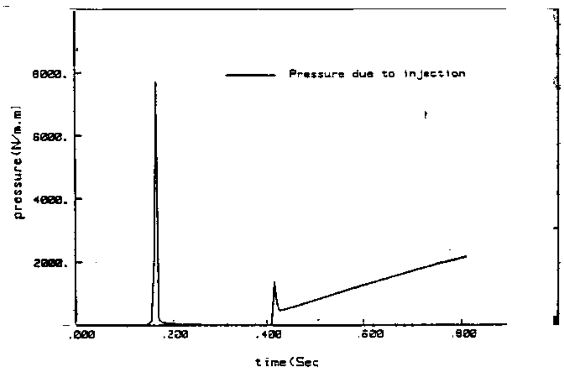

Figure 4. G_{44} : Fluid pressure due to a unit rate of fluid injection.

into the medium at the point (0,0,0) and the resulting horizontal displacement, G_{14}, and the fluid pressure, G_{44}, were evaluated at (0.5,0,0)m.

The velocities of the three waves in this medium are approximately $\frac{1}{f_1} = 3.06, \frac{1}{g_1} = 1.20$ and $\frac{1}{h_1} = 1.68$m/s. Therefore, the three waves arrive at the observation point, i.e., (0.5,0,0), at 0.163, 0.41 and 0.3s. In locating the arrival of the waves in Figures 1 to 4 one should note that, out of the four plotted components of G_{ij}, only G_{11} (Figure 1) contains the three waves. The rest (Figures 2, 3 and 4) contain the two dilatational waves only.

It is also instructive to examine the values of dissipation factors in the three waves. These factors for the two dilatational waves and the shear wave are $f_2 = 0.0063$, $g_2 = 8.8$ and $h_2 = 0.54$, respectively. The large value of g_2, compared to considerably smaller values of f_2 and h_2, is an indication of the highly dissipative nature of the $P2$-wave. As permeability of the medium decreases, the dissipation plays a more pronounced role in suppressing the $P2$-wave, to the extent that this wave can hardly be detected beyond a close vicinity of the source.

An interesting feature displayed by these figures is the interaction between the two phenomena of wave propagation and diffusion. These figures, especially Figures 3 and 4, vividly show that the diffusion starts right after the arrival of the $P2$-wave. In other words, no diffusion takes place before the arrivial of this wave. To demonstrate the significance of diffusion in this problem the same fundamental solutions of Figures 2, 3 and 4 are replotted for larger times in Figures 5, 6, and 7, respectively. Similar conclusions, regarding the two interacting phenomena, apply to generalized thermoelasiticty. Especially, the finite speed of diffusion propagation, which could not be accounted for in classical theories of dynamic thermoelasticity, is noteworthy.

Finally, in order to examine the accuracy of the approximate transient solutions, the same fundamental solution shown in Figure 4, i.e., G_{44}, is obtained by the analytical expression and plotted in Figure 8 against the corresponding numerical result. The properties of the medium and the distance to the source in this comparison are those described before, except that the permeability has in-

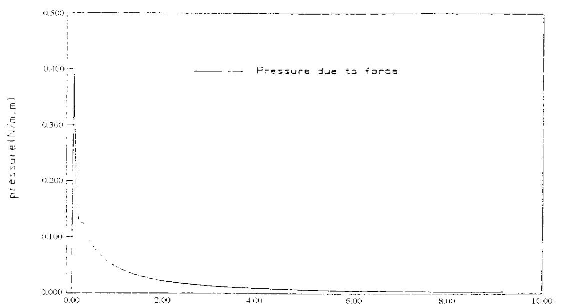

Figure 5. G_{41} : Fluid pressure due to a unit horizontal force.

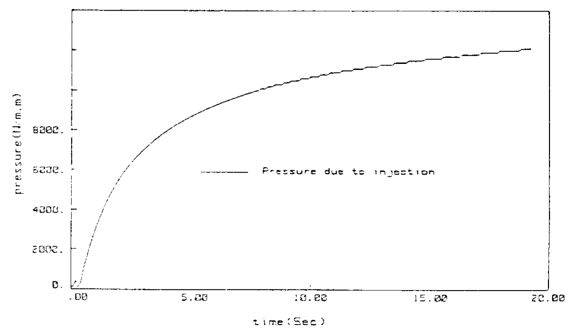

Figure 7. G_{44} : Fluid pressure due to a unit rate of fluid injection.

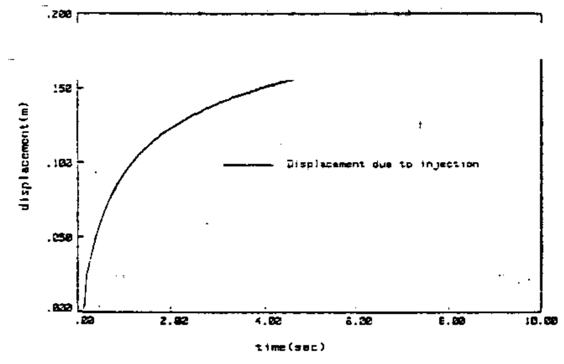

Figure 6. G_{14} : Horizontal solid displacement due to a unit rate of fluid injection.

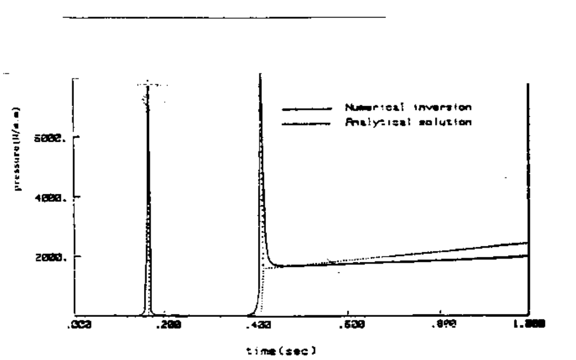

Figure 8. Comparison between the approximate analytical and numerical solutions for G_{44}.

creased such that $b = 2 \times 10^4 \, Ns/m^4$. This comparison, and similar comparisons concerning other fundamental solutions, suggest that the analytical expressions can capture the essential characteristics of the problem and can produce fairly accurate results. However, as expected, the accuracy of these expressions gradually deteriorates as the permeability decreases and as time increases.

CONCLUSION

Fundamental solutions have been derived in the Laplace transform domain for Biot's theory of dynamic poroelasticity. The displacement and pore pressure response is obtained for both point forces and mass sources acting in a domain of infinite extent. Via an analogy, this solution is found to be equally applicable to generalized dynamic thermoelasticity. The correspondence is exact, except that the term associated with the fluid mass density is zero in the latter theory. Thus, displacements and temperatures resulting from the applacation of point forces and heat sources can also be determined based upon the present work.

In light of the complicated form of the fundamental solutions, the inverse transform cannot be completed analytically. Instead, series approximations are introduced to obtain expressions that are more readily inverted. The resulting time domain formulae are valid for the short time response of media with large permeability. One transverse and two longitudinal waves appear in these solutions. All are damped, however the second longitudinal wave is highly dissipative. Additionally, all disturbances propagate with finite velocity, thus eliminating the unrealistic consequence of classical thermoelasticity regarding the instantaneous effects of heat sources applied at a distance. Several numerical results are provided, within the context of poroelasticity, to illustrate the general overall behavior and to validate the approximations.

APPENDIX

Coefficients in the fundamental solutions

$$a_1 = m\left(\frac{1}{M_p} + \frac{\beta^2}{\lambda + 2\mu}\right) + \frac{\rho - 2\beta_{\rho f}}{\lambda + 2\mu}$$

$$a_2 = m\left(\frac{1}{M_p} + \frac{\beta^2}{\lambda + 2\mu}\right)$$

$$b_1 = a_1^2 - \frac{4m}{M_p}\frac{1}{\lambda + 2\mu}\left(\rho - \frac{\rho_f^2}{m}\right)$$

$$b_2 = 2a_1a_2 - \frac{4m}{M_p}\frac{\rho}{\lambda + 2\mu}$$

$$b_3 = a_2^2$$

$$d_0 = \frac{1}{2}\left(a_1 - b_1^{1/2}\right)$$

$$d_1 = \frac{1}{2}\left(a_2 - \frac{1}{b} b_2 b_1^{-1/2}\right) b m^{-1}$$

$$d_2 = -\frac{1}{4} b_1^{-3/2}\left(b_1 b_3 - \frac{1}{4} b_2^2\right) b^2 m^{-2}$$

$$d_3 = \frac{1}{8} b_2 b_1^{-5/2}\left(b_1 b_3 - \frac{1}{4} b_2^2\right) b^3 m^{-3}$$

$$d_4 = \frac{1}{16} b_1^{-7/2}\left(b_1 b_3 - \frac{1}{4} b_2^2\right)\left(b_1 b_3 - \frac{5}{4} b_2^2\right) b^4 m^{-4}$$

$$d_5 = \frac{1}{2}\left(a_1 + b_1^{1/2}\right)$$

$$d_6 = \frac{1}{2}\left(a_2 + \frac{1}{2} b_2 b_2^{-1/2}\right) b m^{-1}$$

$$d_7 = -d_2 \quad , \quad d_8 = -d_3 \quad , \quad d_9 = -d_4$$

$$d_{10} = \frac{1}{\mu}\left(\rho - \frac{\rho_f^2}{m}\right)$$

$$d_{11} = \frac{b}{\mu}\left(\rho - \frac{\rho_f^2}{m}\right)$$

$$d_{12} = -d_{11} b m^{-1}$$

$$d_{13} = d_{11} b^2 m^{-2}$$

$$d_{14} = -d_{11} b^3 m^{-3}$$

$$f_1 = d_0^{1/2}$$

$$f_2 = \frac{1}{2} d_1 d_0^{-1/2}$$

$$f_3 = \frac{1}{2} d_0^{-3/2}\left(d_0 d_2 - \frac{1}{4} d_1^2\right)$$

$$f_4 = \frac{1}{2} d_0^{-5/2}\left(d_0^2 d_3 - \frac{1}{2} d_0 d_1 d_2 + \frac{1}{8} d_1^3\right)$$

$$f_5 = \frac{1}{8} d_0^{-7/2}\left(4 d_4 d_0^3 - 2 d_0^2 d_1 d_3 - d_0^2 d_2^2 + \frac{3}{2} d_0 d_2^2 d_2 - \frac{5}{16} d_1^4\right)$$

$g_k = f_k$, replacing d_{k-1} with d_{k+4}; $k = 1, 5$

$h_k = f_k$, replacing d_{k-1} with d_{k+9}; $k = 1, 5$

$\varepsilon_k = d_k - d_{k+5}$; $k = 0, 4$

$$l_0 = e_0^{-1}$$

$$l_1 = -e_1 e_0^{-2}$$

$$l_2 = e_0^{-2}\left(e_1^2 e_0^{-1} - e_2\right)$$

$$l_3 = e_0^{-2}\left[e_1 e_2 e_0^{-1} - e_3 - e_1 e_0^{-1}\left(e_1^2 e_0^{-1} - e_2\right)\right]$$

$$l_4 = \frac{1}{2} e_0^{-3}\left[e_2^2 + e_1 e_3 - 2e_0 e_4 - e_2 e_0^{-1}\left(e_1^2 - e_0 e_2\right) + 2e_1^2 e_0^{-2}\left(e_1^2 - e_0 e_2\right) - 3e_1 e_0^{-1}\left(e_1 e_2 - e_0 e_3\right)\right]$$

$$d'_k = d_k - \frac{\mu}{\text{[illegible]}} d_{k+10}; \quad k = 0, 4$$

$$d'_{k+5} = d_{k+5} - \frac{\lambda}{\lambda+2\mu} d_{k+10}; \quad k=0,4$$

$$q_o = d'_o l_o$$

$$q_k = \sum_{l=0}^{k} d'_l l_{k-1} + \frac{b}{m} \sum_{l=0}^{k-1} d'_l l_{k-l-1}; \quad k=1,4$$

$q_{k+5} = q_k$, replacing d'_k with d'_{k+5}; $k=0,4$

$$a_3 = \beta m - \rho_f$$

$$a_4 = \beta b$$

$$f'_2 = f_2 + \frac{1}{r}$$

$$A_1 = a_3 l_0 f_1$$

$$A_2 = a_3(l_1 f_1 + l_0 f'_2) + a_4 l_0 f_1$$

$$A_3 = a_3(l_2 f_1 + l_1 f'_2 + l_0 f_3) + a_4(l_1 f_1 + l_0 f'_2)$$

$$A_4 = a_3(l_3 f_1 + l_2 f'_2 + l_1 f_3 + l_0 f_4) + a_4(l_2 f_1 + l_1 f'_2 + l_0 f_3)$$

$$A_5 = a_3(l_4 f_1 + l_3 f'_2 + l_2 f_3 + l_1 f_4 + l_0 f_5) + a_4(l_3 f_1 + l_2 f'_2 + l_1 f_3 + l_0 f_4)$$

$B_1 - B_5 = A_1 - A_5$, replacing f_k with g_k; and f'_2 with $g'_2 = g_2 + \frac{1}{r}$

$l'_k = l_k$, replacing e_k with d_k; $k=0,4$

$l''_k = l_k$, replacing e_k with d_{k+5}; $k=0,4$

$$c_0 = 1 - \frac{m}{M_p} l'_o$$

$$c_k = \frac{1}{M_p}(m l'_k + b l'_{k-1}), \quad k=1,4$$

$c_{k+5} = c_k$ replacing l'_k with l''_k; $k=0,4$

$$m_o = c_o l_o$$

$$m_k = \sum_{l=0}^{k} c_l l_{k-l}; \quad k=1,4$$

$m_{k+5} = m_k$, replacing c_k with c_{k+5}; $k=0,4$

$m_{k+10} = l_k$, replacing e_k with d_{k+10}

$$P^0_{ij} = 3x_i x_j r^{-5} - \delta_{ij} r^{-3}$$

$$P^1_{ij} = 3x_i x_j r^{-4} - \delta_{ij} r^{-2}$$

$$P^2_{ij} = x_i x_j r^{-3}$$

$$N^0_{ij} = P^2_{ij} d_0$$

$$N^1_{ij} = P^1_{ij} f_1 + P^2_{ij} d_1$$

$$N^2_{ij} = P^0_{ij} + P^1_{ij} f_2 + P^2_{ij} d_2$$

$$N^3_{ij} = P^1_{ij} f_3 + P^2_{ij} d_3$$

$$N^4_{ij} = P^1_{ij} f_4 + P^2_{ij} d_4$$

$N^{k+5}_{ij} = N^k_{ij}$, replacing d_k by d_{k+5} and f_k by g_k; $k=0,4$

$N^{k+10}_{ij} = N_k$, replacing d_k by d_{k+10} and f_k by h_k; $k=0,4$

$$C^k_{ij} = \sum_{l=0}^{k-1} m_l N^{k-l-1}_{ij}; \quad k=1,4$$

$$C^k_{ij} = \sum_{l=5}^{k} m_l N^{k-l}_{ij}; \quad k=5,8$$

$$C^k_{ij} = \sum_{l=10}^{k-1} m_l N^{k-l-1}_{ij}; \quad k=11,14$$

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