IJE TRANSACTIONS C: Aspects Vol. 33, No. 3, (March 2020) 439-447

International Journal of Engineering

Journal Homepage: www.ije.ir

Coordination of Pricing and Order Quantity for Two Replaceable and Seasonal Products

N. Rasouli[a,b], M. R. Amin-Naseri*[a], I. Nakhai Kamalabadi[c], A. Hosseinzadeh Kashan[a]

[a] *Department of Industrial and System Engineering, Tabriat Modares University, Iran*
[b] *Group for Research in Decision Analysis (GERAD), HEC Montreal University, Canada*
[c] *Department of Industrial Engineering, University of Kurdistan, Kurdistan, Iran*

PAPER INFO

Paper history:
Received 26 November 2019
Received in revised form 16 December 2019
Accepted 22 Janurary 2020

Keywords:
Pricing
Order Quantity
Seasonal Products
Substitution
Karush Kuhn Tucker Conditions

ABSTRACT

This paper deals with the coordination of pricing and order quantity decisions for two seasonal and substitutable goods in one firm. We assume that the customers are price sensitive and they are willing to buy the cheaper products, which is known as one way and customers-based price driven substitution. First, a mathematical model is developed for one firm, which contains two replaceable products considering seasonality. The model aims to maximize the profit by determining optimal dynamic prices, order quantities and the number of periods for both of the products. Then, we show that the objective function is strictly concave of price and has a unique maximum solution. Next, an exact algorithm based on the Karush Kuhn Tucker (KKT) conditions is presented to determine the optimal decisions. Finally, a numerical example accompanied by sensitivity analysis on key parameters is developed to illustrate the efficiency of solution procedure and the algorithm.

doi*: 10.5829/ije.2020.33.03c.08*

NOMENCLATURE

Indices		g_i	Time coefficient for demand of product i
i	Products (i=1,2)	$I_{i,j}(t)$	Inventory levels at time t of period j for product i
j	Periods (j=1,2,…,n)	h_i	Inventory holding cost per unit of product i
t	time	s	Unit price setting cost
Parameters		c_i	Unit purchasing cost of product i
T	Time horizon (sale season)	v_i	Transportation cost of product i
N_{max}	Maximum number of periods	π	Total profit of the firm
M	Length of each period ($M=\frac{T}{n}$)	**Decision Variables**	
$d_{t,i,j}$	Demand of product i at time t and period j.	$p_{i,j}$	Unit selling price of product i during period j ($p_{i,j} > c_i$)
α_i	Market potential of products i	q_i	Order quantity of product i
β_i	Price coefficient for demand of product i	n	Number of periods/ number of price settings ($n \le N_{max}$)

1. INTRODUCTION

Pricing or order quantity in the supply chain has been considered separately for a long time [1, 2, 3], while nowadays dynamic pricing and inventory decisions are mentioned simultaneously by many researchers as the most important factors towards the success of a business. Dynamic pricing is applied by the researchers for the products whose demand and price change during the time like perishable products [4, 5] and seasonal products. One of the researches in the field of pricing for seasonal goods is done by Aviv and Pazgal. They used Nash game to find the equilibrium solutions, considering inventory dependent discount and fixed discount [6]. Another research, which studied the effect of spot and forward purchases on the pricing and order quantity decisions for

*Corresponding Author Email: *amin_nas@modares.ac.ir* (M. R. Amin-Naseri)

Please cite this article as: N. Rasouli, M. R. Amin-Naseri, I. Nakhai Kamalabadi, A. Hosseinzadeh Kashan, Coordination of Pricing and Order Quantity for Two Replaceable and Seasonal Products, International Journal of Engineering (IJE), IJE TRANSACTIONS C: Aspects Vol. 33, No. 3, (March 2020) 439-447

the seasonal products, has been done [7]. In their research, shortages are not allowed and the demand functions for both types of sale are time and price dependent.

Substitution is another feature that we considered in this paper, which can be customer based or manufacturer based in general and on the other hand it can be price based or inventory based [8]. Inventory based substitution related to the cases in which the customers substitute the out of stock products with the available ones [9]. Whereas, in the case of price based substitution, the customers are willing to buy the cheaper products so that they replace the expensive goods with the similar less expensive ones [10]. Rasouli and Nakhai [11] have studied dynamic pricing and order quantity for a seasonal good considering substitution. In their research, the aim is to maximize the profit of the retailer by determining price and order size for one of the products considering asymmetrical substitution in which just a fraction of the customers substitute the goods while the others cancel the orders when they face with expensive commodities.

Wang and Huang [12] proposed a model toward pricing and inventory control for seasonal deteriorating products with time dependent demand and fixed selling season. Single period pricing and inventory management in the presence of strategic customers is reviewed by Du et al. [13]. They studied the effect of risk preference and decreasing value of strategic customers on the variables (price and order quantity). Finally, they examine the effect of re-payment contract on the reduction of this behavior. Rabbani et al. [14] studied the model of perishable goods with quality and physical quantity deterioration. The model is aimed to find optimum price, discount rate and replenishment cycle. Naimi Sadigh et al. [15] also developed a model of pricing, order quantity and marketing in a three level supply chain with several suppliers, one manufacturer and several retailers which is solved by Nash equilibrium. The competitive model between two supply chains consisting of a manufacturer and retailer with replaceable goods for a real problem in a food industry was developed by Aminnaseri and Azari Khojasteh [16]. Bian et al. [17] have developed a model for determining competitive price between the retailers of Apple and Samsung products consisting of one manufacturer and one retailer. The demand function is depended on price, individual quality valuation and quality attributes. Results showed that not considering stock out-base substitution leads to incompatibility in supply-demand and causing the lower profit. Taleizadeh and Baghban [18] developed a model of pricing and lot sizing for deteriorating items under group dispatching under two scenarios. In the first scenario, the deterioration rate is price sensitive and demand is time and price sensitive. In the second scenario the deterioration rate is constant and demand is depended on time and price.

Some of the researchers developed their models in the stochastic environment. In this regard, Karakul and Chan [19] considered a joint pricing and inventory control for one-way substitution products with stochastic price dependent demand. They modeled the demand in an additive stochastic form and assumed that the stochastic term for the high-grade product (new product) follows a continuous distribution function and the term for the low grade product (existing product) follows a discrete distribution function. Then, other researcher [20] developed this model to the case demand distribution for both grades, which follow a continuous distribution function. Also, Soleimani et al. [21] developed a model of pricing and inventory control in a two level supply chain composed of a retailer and a wholesaler under the condition of demand disruption. Using a stochastic inventory system in pricing model was considered by Li et al. [1]. In this paper dynamic pricing and periodic ordering for perishable products with stock dependent demand is modeled. Stochastic demands were considered by other researchers [22, 23]. Pricing and inventory management are also done in multi-level supply chain [24-27]. As in this paper, we considered one level, we did not get in to the details for the literature of multi-level supply chain.

The aforementioned papers primly studied the models of pricing and inventory control for seasonal or substitutable goods, separately; while, in the competitive markets, most of the seasonal goods are replaceable by price sensitive customers, so the demand is linked to the ability of customers to substitute the products. In other words, various brands or various types of a product, lead the price sensitive customers to be able to decide between the similar products and choose the cheapest one. This happening causes substitution, which is undeniable in the real world. However, most of the papers in the literature ignore it; in which the results get away from reality. The aforementioned discussion becomes more important for seasonal products, because of the limited sale period of time.

In the reviewed literature, one of the researcher [11] has investigated seasonality and substitution simultaneously. However, it was assumed two replaceable products with given price and order size for one of them and they tried to find the variables just for one of the products. While, due to the dependency of price and order size of one product to the price and order size of replaceable product. It is better not to consider given price and order size for one of the products, but to determine the variables for both of the products simultaneously. Therefore, there is a gap in this field, which motivated us to focus on it because of the importance of subject. Hence, in this paper, we examine joint pricing and order quantity for two seasonal goods with customer-based price-driven substitution, simultaneously. The demand function is based on time, price of product and the price differences of goods. The major objective is to determine the optimal dynamic prices, order size and the optimal number of selling periods for both of the products simultaneously, such that the total profit of the firm is maximized.

The rest of the article is organized as follows: In section 2, the notations and assumptions of the proposed model are presented. In section 3, a mathematical model is presented based on the mentioned settings. Then we prove that for any given number of price settings, the objective function is a strictly concave function of selling prices. In section 4, we present an exact algorithm to find the optimal prices, order quantities, and number of periods for each product. Moreover, in section 5 in order to evaluate the accuracy of proposed model accomplished with the performance analyzing of the proposed solution approach a numerical example has been illustrated followed by conclusion and some directions as future studies in section 6.

2. MATHEMATICAL MODELING

We consider a firm selling two seasonal and replaceable products during time horizon *T*. The problem is aimed to determine the price and order quantity for both of the products by maximizing the profit of the firm considering the following assumptions:

- ❖ Demand of each product is dependent on time, price of the product and the price of replaceable product.
- ❖ The time horizon (*T*) is divided to *n* periods.
- ❖ The price of each product changes *n* times (at each period).
- ❖ There are two products, which are substitutable by price sensitive customers.
- ❖ The ordering is done only at the beginning of the sale season.
- ❖ Shortages and backlogs are not allowed.

For mathematical modeling, first we define the demand function according to literature [7, 11] as discussed below. About the contribution of our model, we can refer to seasonality and substitution, which are ignored in most of literature which is considered unknown price ($p_{i,j}$, $p_{w,j}$) and order quantity (q_i, q_w) for both of the replaceable products. Our contribution is not only considering seasonality and replacing feature, but also determining the optimal prices and order quantities for both of the replaceable products, simultaneously.

$$d_{t,i,j} = \alpha_i e^{-g_i t} - \beta_i p_{i,j} - L\left(p_{i,j} - p_{w,j}\right) \qquad (1)$$
$$i, w = 1,2,\ w \neq i, (j-1)M \leq t \leq jM,\ 1 \leq j \leq n$$

The first term of the above function shows the seasonality, so that demand decreases during time. The second terms shows the dependency of demand to the price of product, which is usual and obvious according to the literature. Finally, the last term is related to the substitution, so that demand is affected by price differences in the negative way.

As shown in Figure 1, the inventory starts at $\boldsymbol{q_i}$ and decreases exponentially over time and it gets to zero at the end of the season (T). According to this figure, the inventory behavior changes from period to period, which is because of assumption (3), mentioning that the prices change at each period. Using this figure and demand function, the inventory equation is written as follows:

$$\mathrm{I}_{t,i,j} = q_i - \sum_{k=1}^{j-1} \left(\int_{(k-1)M}^{kM} d_{t,i,k}\, dt\right) - \int_{(j-1)M}^{t} d_{t.i.j}\, d \qquad (2)$$
$$i = 1,2,\ (j-1)M \leq t \leq jM, 1 \leq j \leq n$$

The first term of above equation shows the total quantity, which is ordered at the beginning of the sale season. The second term is related to the cumulative demand from the beginning of season (t=0) up to the end of (*j-1*) th period and the last term shows the total demand from the beginning of period *j* up to time *t*.

Based on the assumptions, shortages are not allowed and the ordering is done at the beginning of the sale season. Hence, the order quantity is equal to the cumulative demand over the season as follows:

$$q_i = \sum_{j=1}^{n} \int_{(j-1)M}^{jM} d_{t,i,j}\, dt \qquad i = 1,2,\ 1 \leq j \leq n \qquad (3)$$

According to the above-mentioned equations and considering the maximization behavior of profit function, we have:

$$\pi = \sum_{i=1}^{2} \sum_{j=1}^{n} \left(\left(p_{i,j} - c_i\right) \int_{(j-1)M}^{jM} d_{t,i,j}\, dt \right.$$
$$\left. - \sum_{i=1}^{2} \sum_{j=1}^{n} h_i \int_{(j-1)M}^{jM} I_{t,i,j}\, dt \ - ns - \sum_{i=1}^{2} \left(q_i v_i\right) \right. \qquad (4)$$

St:

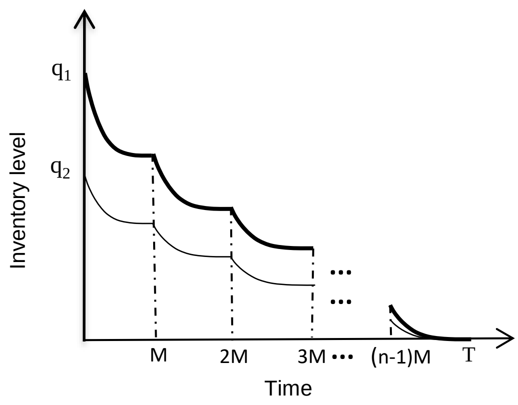

Figure 1. The inventory level of products

$$d_{Mj,i,j} \geq 0 \longrightarrow p_{i,j} \leq \frac{\alpha_i e^{-g_i Mj} + Lp_{w,j}}{\beta_i + L} \tag{5}$$

$i, w = 1,2,\ w \neq i,\ 1 \leq j \leq n$

$$n \leq N_{max} \tag{6}$$

The first term of objective function is related to the sale profit, which is obtained by selling the products. The second and third terms indicate the inventory cost and price setting cost, respectively. Finally, the last term shows the transportation cost of the products to the firm. Equations (5) and (6) indicate the constraints of the problem, so that the former one is related to the non-negativity of demand and the latter one indicates the upper bound of periods. About Equation (5), as the demand is decreasing during time, it reaches to the minimum amount at the end of each period. Consequently, instead of $d_{t,i,j} \geq 0$, we consider $d_{Mj,i,j} \geq 0$ which is related to the demand at the end of each period.

3. SOLUTION METHOD

In this paper we are determine the prices, order quantities, and number of periods in order to maximize the total profit of the firm. Order quantity for each product is obtained by Equation (3) and now we want to determine the other variables. For simplicity, we considered the problem for any given number of periods (n) and then maximize it based on $n \leq N_{\max}$. In other words, first we solved the model according to Equations (4) and (5) for any n which is denoted by Z_n , then we repeat the solving for any $n \leq N_{\max}$ and choose the maximum profit.

Since Z_n is a multi-variable and constrained nonlinear problem, the Karush-Kuhn-Tucker (KKT) conditions are necessary for the optimal solutions. By converting the Equation (5) to equality constraint, the Lagrangian function of problem Z_n is as follows, where $\boldsymbol{\lambda_{i,j}}$ is the Lagrangian coefficient, which is non-negative and $\boldsymbol{\phi^2_{i,j}}$ is the slack amount to equating the constraint to zero:

$$\mathrm{L} = \pi - \sum_{i=1}^{2} \sum_{j=1}^{n} \lambda_{i,j} \left(p_{i,j} - \frac{\alpha_i e^{-g_i \mathrm{M}j} + Lp_{w,j}}{\beta_i + L} + \phi^2_{i,j} \right) \tag{7}$$

$i, w = 1,2,\ w \neq i,\ 1 \leq j \leq n$

The KKT necessary conditions are obtained by taking the partial derivatives of Lagrangian function with respect to $p_{i,j}$, $\lambda_{i,j}$, and $\phi_{i,j}$ for any j as follows:

$$\frac{\partial L}{\partial p_{1,j}} = \frac{\partial F}{\partial p_{1,j}} - \lambda_{1,j} + \frac{\lambda_{2,j} L}{\beta_2 + L} = 0 \tag{8}$$

$$\frac{\partial L}{\partial p_{2,j}} = \frac{\partial F}{\partial p_{2,j}} - \lambda_{2,j} + \frac{\lambda_{1,j} L}{\beta_1 + L} = 0 \tag{9}$$

$$\frac{\partial L}{\partial \lambda_{1,j}} = \left(p_{1,j} - \frac{\alpha_1 e^{-g_1 Mj} + Lp_{2,j}}{\beta_1 + L} + \phi^2_{1,j} \right) \tag{10}$$

$$\frac{\partial L}{\partial \lambda_{2,j}} = \left(p_{2,j} - \frac{\alpha_2 e^{-g_2 Mj} + Lp_{1,j}}{\beta_2 + L} + \phi^2_{2,j} \right) \tag{11}$$

$$\frac{\partial L}{\partial \phi_{1,j}} = -2\lambda_{1,j}\phi_{1,j} = 0 \tag{12}$$

$$\frac{\partial L}{\partial \phi_{2,j}} = -2\lambda_{2,j}\phi_{2,j} = 0 \tag{13}$$

These necessary conditions are sufficient if the maximizing profit function (Equation (4)) is concave. For this reason, we have the following theorem.

Theorem 1. *For any given n,* $\boldsymbol{\pi(n, p_{i,j})}$ *is a concave function of* $\boldsymbol{p_{i,j}}$.

Proof. Please see Appendix A.

Now in order to determine the optimal solution, we construct the first derivation of objective function with respect to prices as follow:

$$\begin{aligned}\frac{\partial F}{\partial p_{i,j}} = & -LM(c_w + v_w) + \frac{1}{2}(1-2j)LM^2(h_i - h_w) \\ & + L(-j+n)M^2(h_i - h_w) - LnM^2 h_w + 2LM\mathrm{p}_{w,j} \\ & -2M\mathrm{p}_{i,j}(L+\beta_i) + \frac{e^{-jMg_i}(-1+e^{Mg_i})\alpha_i}{g_i} - \frac{1}{2}M^2 h_i \beta_i + \\ & M(L+\beta_i)(c_i+v_i) + nM^2 h_i (\mathrm{L}+\beta_i)\end{aligned} \tag{14}$$

Now by equating the above equation to zero we have $p_{i,j}$ based on $p_{w,j}$ for the result of substitution. Therefore, by solving system of equations with two variables, the following equation for any $i, w = 1,2,\ w \neq i,\ 1 \leq j \leq n$ is obtained:

$$\begin{aligned}p_{i,j}(1) = & \frac{c_i + v_i}{2} + \frac{1}{4(L\beta_i + \beta_w(L+\beta_i))} \Big(\frac{2e^{-jMg_i}(-1+e^{Mg_i})\alpha_i(L+\beta_w)}{Mg_i} \\ & \frac{2e^{-jMg_w}(-1+e^{Mg_w})\alpha_w L}{Mg_w} + (L(2M(-1+2j-n)h_w + M(1- \\ & 4j+4n)h_i)\beta_w + (M(-1+2n)h_i)(L+\beta_w) + \beta_i)\end{aligned} \tag{15}$$

Then, we denote the following two consecutive equations for any i , w ($w \neq i$), and j:

$$p_{i,j}(2) = \frac{\alpha_i e^{-g_i jM} + Lp_{w,j}}{\beta_i + L} \tag{16}$$

$$\begin{aligned}x_{i,j} = & Mc_i(L+\beta_i) + \frac{1}{2g_i}\Big(e^{-jMg_i}\Big(2(-1+e^{Mg_i} - \\ & 2Mg_i)\alpha_i - e^{jMg_i}Mg_i\big(L(2c_w + M(-1+4j-4n)(h_i \\ & -h_w) - 2v_i + 2v_w) + \big((M-2Mn)h_i - 2v_i\big)\beta_i\big)\Big)\Big)\end{aligned} \tag{17}$$

$p_{i,j}(2)$ is the upper bound of price according to constraint (5) and $x_{i,j}$ is obtained by substituting $p_{i,j} = p_{i,j}(2)$ into Equation (14).

For any number of periods (n), the objective function is strictly concave (see Theorem 1). Thus, the optimal solution is unique. In order to find the optimal solution, we develop the following Lemmas:

Lemma 1. *For fixed n, if* $\boldsymbol{x_{1,j}}$*>0 and* $\boldsymbol{x_{2,j} \leq 0}$*, we have two states:*

1) If $\boldsymbol{x_{1,j} \leq -\frac{(L+\beta_1)}{L} x_{2,j}}$ then, $\boldsymbol{p_{1,j} = p_{1,j}(2)}$ *and* $\boldsymbol{p_{2,j} = p_{2,j}(1)}$ *satisfy the KKT necessary conditions.*

2) If $x_{1\mathrm{j}} > -\frac{(L+\beta_1)}{L}x_{2\mathrm{j}}$ then, $p_{1,j} = p_{1,j}(2)$ *and* $p_{2,j} = p_{2,j}(2)$ *satisfy the KKT necessary conditions.*

Proof.

1) Let $\phi_{1,j}$,$\lambda_{2,j}$=0. According to Equation (10) the value of $p_{1,\mathrm{j}}$ is obtained, which is equal to Equation (16), shown as $p_{1,j}(2)$. Substituting $p_{1,\mathrm{j}}$ into Equation (8) yields $\lambda_{1,j} = \frac{\partial F}{\partial p_{1,\mathrm{j}}} = x_{1,j} > 0$ and substituting $p_{1,\mathrm{j}}$ and $\lambda_{1,j}$ into Equation (9) gives $p_{2,\mathrm{j}}$ equal to $p_{2,\mathrm{j}}(1)$. Replacing $p_{2,\mathrm{j}}$ into Equation (11) yields ${\phi_{2,j}}^2 = \frac{1}{2(L+\beta_2)M}(-x_{2,\mathrm{j}} - \frac{L}{(L+\beta_1)}x_{1,\mathrm{j}})$>0 (positivity of ${\phi_{2,j}}^2$ obtains from the condition $x_{1,\mathrm{j}} \leq -\frac{(L+\beta_1)}{L}x_{2,\mathrm{j}}$. Therefore, $p_{1,\mathrm{j}}(2)$ and $p_{2,\mathrm{j}}(1)$ are feasible solutions to KKT necessary conditions and are accordingly the unique optimal solution for the problem.

2) Let $\phi_{1,j}$,$\phi_{2,j}$=0. According to Equations (10) and (11), the values of $p_{1\mathrm{j}}$ and $p_{1\mathrm{j}}$ are obtained, which are equal to Equation (16), shown as $p_{1,j}(2)$ and $p_{2,j}(2)$, respectively. Substituting $p_{1,\mathrm{j}}$ and $p_{2,\mathrm{j}}$ into Equations (8) and (9) yields: $\lambda_{1,j}$=$\frac{(L+\beta_1)(Lx_{1,\mathrm{j}}+Lx_{2,\mathrm{j}}+x_{1,\mathrm{j}}\beta_2)}{L\beta_1+L\beta_2+\beta_1\beta_2}$ >0 and $\lambda_{2,j}$=$\frac{(L+\beta_2)(Lx_{1,\mathrm{j}}+x_{2,\mathrm{j}}(L+\beta_1))}{L\beta_1+L\beta_2+\beta_1\beta_2}$ >0. Positivity of $\lambda_{1,j}$ and $\lambda_{2,j}$ comes from the condition $x_{1,\mathrm{j}} > -\frac{(L+\beta_1)}{L}x_{2,\mathrm{j}}$.

Lemma 2. *For fixed n, if* $\boldsymbol{x_{1,j}}$ *≤0 and* $\boldsymbol{x_{2,j} > 0}$, *we have two states:*

1) If $\boldsymbol{x_{2,\mathrm{j}} \leq -\frac{(L+\beta_2)}{L}x_{1,\mathrm{j}}}$ then, $\boldsymbol{p_{1,j} = p_{1,j}(1)}$ *and* $\boldsymbol{p_{2,j} = p_{2,j}(2)}$ *satisfy the KKT necessary conditions.*

2) If $x_{2,\mathrm{j}} > -\frac{(L+\beta_2)}{L}x_{1,\mathrm{j}}$ then, $p_{1,j} = p_{1,j}(2)$ *and* $p_{2,j} = p_{2,j}(2)$ *satisfy the KKT necessary conditions.*

Proof.

1) Let $\phi_{2,j}$, $\lambda_{1,j}$=0. According to Equation (11) the value of $p_{2,\mathrm{j}}$ is obtained, which is equal to Equation (16), shown as $p_{2,j}(2)$. Substituting $p_{2,\mathrm{j}}$ into (9) yields $\lambda_{2,j} = \frac{\partial F}{\partial p_{2.\mathrm{j}}} = x_{2,j} > 0$ and substituting $p_{2,\mathrm{j}}$ and $\lambda_{2,j}$ into Equation (8) gives $p_{1,\mathrm{j}}$ equal to $p_{1,\mathrm{j}}(1)$. Replacing $p_{1,\mathrm{j}}$ into Equation (10) yields ${\phi_{1,j}}^2 = \frac{1}{2(L+\beta_1)M}(-x_{1,\mathrm{j}} - \frac{L}{(L+\beta_2)}x_{2,\mathrm{j}})$>0 (positivity of ${\phi_{1,j}}^2$ obtains from the condition $x_{2,\mathrm{j}} \leq -\frac{(L+\beta_2)}{L}x_{1,\mathrm{j}}$). Therefore, $p_{1,\mathrm{j}}(2)$ and $p_{2,\mathrm{j}}(1)$ are feasible solutions to KKT necessary conditions and are accordingly the unique optimal solution for the problem.

2) Let $\phi_{1,j}$,$\phi_{2,j}$=0 and the proof is same as the proof of Lemma 1, case 2.

Lemma 3. *For fixed n, if* $\boldsymbol{x_{1,j}}$ *≤0 and* $\boldsymbol{x_{2,j} \leq 0}$, $\boldsymbol{p_{1,j} = p_{1,j}(1)}$ *and* $\boldsymbol{p_{2,j} = p_{2,j}(1)}$ *satisfy the KKT necessary conditions.*

Proof. Let $\lambda_{1,j}$, $\lambda_{2,j}$=0. According to Equations (8) and (9), the value of $p_{1,\mathrm{j}}$ and $p_{2,\mathrm{j}}$ will be obtained, which are equal to $p_{1,\mathrm{j}}(1)$ and $p_{2,\mathrm{j}}(1)$. Replacing $p_{1,\mathrm{j}}$ and $p_{2,\mathrm{j}}$ into Equations (10) and (11), yields ${\phi_{1,j}}^2 = -\frac{x_{1,j}}{2M(L+\beta_1)}$>0 and ${\phi_{2,j}}^2 = -\frac{x_{2,j}}{2M(L+\beta_2)} > 0$, respectively. Therefore, $p_{1,\mathrm{j}}(1)$ and $p_{2,\mathrm{j}}(1)$ are feasible solutions to KKT necessary conditions and are accordingly the unique optimal solution for the problem.

Lemma 4. *For fixed n, if* $\boldsymbol{x_{1,j}}$ *>0 and* $\boldsymbol{x_{2,j} > 0}$, $\boldsymbol{p_{1,j} = p_{1,j}(2)}$ *and* $\boldsymbol{p_{2,j} = p_{2,j}(2)}$ *satisfy the KKT necessary conditions.*

Proof. Let $\phi_{1,j}$,$\phi_{2,j}$=0 and the proof is the same as proof of Lemma 1, case 2.

According to the mentioned Lemmas, we can generate a solution procedure to obtain the optimal values of $p_{1,\mathrm{j}}, p_{2,\mathrm{j}}, n, q_1$ and q_2. We start with *n*=2. This is because, *n*=1 is used for the static pricing while we use dynamic pricing.

4. ALGORITHM

1. Start with *j*=1, *n*=2, *N*=2, π^*=0.
2. While $n \leq N_{\max}$ do steps 3-8, else go to step 9.
3. While *j*≤*n* do steps 4-5, else go to 6.
4. Calculate $x_{1,j}$ and $x_{2,j}$.

4.1. If $x_{1,j} > 0$, $x_{2,j} \leq 0$:

4.1.1. If $x_{1,\mathrm{j}} \leq -\frac{(L+\beta_1)}{L}x_{2,\mathrm{j}}$, then use Lemma 1, case 1 to determine $p_{1,j}$ and $p_{2,j}$.

4.1.2. If $x_{1,\mathrm{j}} > -\frac{(L+\beta_1)}{L}x_{2,\mathrm{j}}$, then use Lemma 1, case 2 to determine $p_{1,j}$ and $p_{2,j}$.

4.2. If $x_{1,j} \leq 0$, $x_{2,j} > 0$:

4.2.1. If $x_{2,\mathrm{j}} \leq -\frac{(L+\beta_2)}{L}x_{1,\mathrm{j}}$, then use Lemma 2, case 1 to determine $p_{1,j}$ and $p_{2,j}$.

4.2.2. If $x_{2,\mathrm{j}} > -\frac{(L+\beta_2)}{L}x_{1,\mathrm{j}}$, then use Lemma 2, case 2 to determine $p_{1,j}$ and $p_{2,j}$.

4.3. If $x_{1,j} \leq 0$, $x_{2,j} \leq 0$ use Lemma 3 to determine $p_{1,j}$ and $p_{2,j}$.

4.4. If $x_{1,j} > 0$, $x_{2,j} > 0$ use Lemma 4Lemma to determine $p_{1,j}$ and $p_{2,j}$.

5. Set *j*=*j*+1 and go back to step 3.
6. Calculate $q_1, q_2, F(n, pa_j^*, pb_j^*)$ by Equations (3) and (4).
7. If $\pi > \pi^*$, then let *N*=*n*, $p_{1,j}^* = p_{1,j}$, $p_{2,j}^* = p_{2,j}$, $q_1^* = q_1$, $q_2^* = q_2$, $\pi^* = \pi$.
8. Set *n*=*n*+1 and go back to step 2.
9. Stop.

5. NUMERICAL EXAMPLE

To illustrate the solution procedure, we solve the following numerical example. We consider two types of ice cream, which are seasonal products and replaceable by customers, as a case study for this section. The data are gathered from a data base web site[1]. These data are related to demand and price of ice creams and frozen foods in US, during 1973 to 2019 (June-August). Using these data, we predict the demand parameters by regression method (Appendix B) as bellow. In addition, for value of costs like purchasing cost, holding cost, …, we assume as follows:
$c_1 = \$1.1$, $c_2 = \$1$, T=720 hours (equals to 3 months), $h_\mathrm{i} = 0.0026\$$/per unit time, $h_2 = \$0.0024$/per unit time, s=\$50, $v_1 = v_2 = \$0.001$, $N_{\max}$=4, L= 23.86, α_1 and α_2 are 13.6 and 8.47, β_1 and β_2 are 0.26 and 3.6, respectively. Finally, the amount of g_1 and g_2 are 0.001. The results are shown on Table 1, which are obtained by Mathematica package 8.0.1.

It is clear from Table 1 that the maximum profit (\$5390.97) is obtained by n=2, which means that in the optimal case, the sale season consists of two periods, so that the first period contains the time interval [0, 360] and the second period contains the time interval [360^+,720]. The optimal prices of the products 1, 2 during first period are: $p_{1,1}$=\$3.88 and $p_{2,1}$=\$3.53 and during second period are: $p_{1,2}$=\$3.01 and $p_{2,2}$=\$2.76. Also, the optimal order quantities are: $q_1 = 1182$ and $q_2 = 1350$. Comparing the obtained prices with the real mean prices that are \$2.5 for the first type and \$2.2 for the second type, it is concluded that the optimal prices are more than the real ones. It means that by increasing the prices, the company will earn more profit. Moreover, we can conclude from Table 1 that by increasing the number of periods (*n*), the profit decreases. Since, the concavity of the objective function is proved and the optimal solution is obtained by *n=2*, so it is obvious that by increasing *n (n>2)*, the profit is reduced.

5. 1. Sensitivity Analysis In this section, we want to investigate the impact of main parameters (demand parameters, purchasing cost and holding cost) on the optimal variables and total profit. For this reason, we change the parameters in the range of -30 to +30% by step 15%, and obtain the results, as shown in Table 2. In order to have better comparison, we set *n=3* for the number of periods.

The results show that by increasing the market potential (α_i), the optimal prices and demand are increasing, which causes the increase of profit remarkably. In other words, by increasing the market potential, demand size increases. Hence, by increasing the demand and prices, the total profit increases significantly. Reviewing the results for impact of price coefficient (β_i) indicates that by increasing β_i(increase of price elasticity), the prices, order sizes, and profit decrease. That is to say, when the price elasticity is increasing, the customers become more sensitive to the price; so that the firm should reduce the prices to attract the customers, which leads to the reduction of profit.

In addition, increasing time coefficient (g_i), means that the demand decreases more over time, which leads to price reduction and loss of profit. For holding and purchasing costs, it is concluded from Table 2 that when these costs increase, the firm should raise the price to compensate the costs, which affects the demand to be reduced . Hence, by increasing price and decreasing the demand, the total profit is reduced as shown in Table 2. Comparing the effect of holding and purchasing cost on the profit indicates that the purchasing cost has more impact on total profit; so that by 30% reduction of purchasing cost, the total profit reaches \$7463.49, while for holding cost it is \$7018.64.

At last, the impact of substitution coefficient is investigated on the optimal solution. That way, by decreasing the tendency of customers to replace the products, the price differences of the products become more than before, and consequently the profit increases. Finally, considering $L=0$ (no substitution), causes sigificant increase of profit. In other words, in the case that there is no substitution, the customers buy their favorite products without considering the price differences with the similar products, which causes considerable increase of profit.

TABLE 1. Computational results

n	**2**		**3**			**4**			
j	1	2	1	2	3	1	2	3	4
$p_{1,j}$**(\$)**	3.88	3.01	4.12	3.49	2.98	4.25	3.75	3.32	2.96
$p_{2,j}$**(\$)**	3.53	2.76	3.74	3.20	2.74	3.85	3.42	3.07	2.73
q_1	1182		1109			1115			
q_2	1350		1195			1074			
π(\$)	**5390.97**		5383.77			5317.66			

[1] https://fred.stlouisfed.org

TABLE 2. Sensitivity analysis

$p_{1,j}$(\$)	% of change	$p_{1,j}$(\$)			$p_{2,j}$(\$)			q_1	q_2	π(\$)
		j=1	*j=2*	*j=3*	*j=1*	*j=2*	*j=3*			
α_i	+30	4.94	4.14	3.49	4.50	3.80	3.23	2335	1645	11218.2
	+15	4.53	3.81	3.24	4.12	3.50	2.99	1750	1387	8015.91
	-15	3.70	3.16	2.55	3.36	2.90	2.35	765	838	3453.62
	-30	3.26	2.67	2.10	2.98	2.46	1.93	606	451	2106.53
β_i	+30	3.54	3.01	2.37	3.16	2.70	2.12	891	767	3572.80
	+15	3.79	3.41	3.23	2.94	2.64	2.40	1074	865	4290.89
	-15	4.57	3.84	3.26	4.19	3.56	3.04	1325	1463	7152.85
	-30	5.21	4.35	3.66	4.83	4.06	3.46	1428	1860	9832.31
g_i	+30	4.03	3.27	2.42	3.65	3.00	2.22	885	1164	5010.40
	+15	4.09	3.41	2.80	3.71	3.13	2.57	1025	1179	5239.27
	-15	4.17	3.61	3.14	3.78	3.31	2.90	1241	1169	5762.72
	-30	4.22	3.73	3.32	3.83	3.43	3.07	1606	1248	6238.11
c_i	+30	4.38	3.90	3.48	3.98	3.58	3.21	1289	1155	5159.81
	+15	4.30	3.81	3.40	3.90	3.50	3.14	1419	1239	5689.14
	-15	4.13	3.65	3.23	3.75	3.35	2.99	1737	1341	6848.96
	-30	4.05	3.57	3.15	3.68	3.28	2.92	1924	1351	7463.49
h_i	+30	4.43	3.94	3.53	4.04	3.64	3.25	1395	902	5622.72
	+15	4.35	3.85	3.41	3.92	3.52	3.15	1127	1482	5928.55
	-15	4.10	3.61	3.20	3.71	3.31	2.96	1686	1494	6619.37
	-30	3.98	3.49	3.08	3.59	3.19	2.85	1765	1739	7018.64
L	+30	4.17	3.69	3.28	3.83	3.43	3.09	1117	1728	6099.91
	+15	4.19	3.71	3.30	3.83	3.43	3.08	1332	1519	6159.64
	-15	4.25	3.76	3.34	3.83	3.42	3.05	1873	1002	6362.51
	-30	4.31	3.81	3.38	3.82	3.42	3.03	2045	847	6518.53
	L=0	24.40	21.68	18.53	1.98	1.68	1.42	3660	402	74052.6

6. CONCLUSION

In this paper, we developed mathematical modeling for dynamic pricing and order quantity for two seasonal and replaceable products, which are belonged to one firm and are sold in two retailers. Most of the researches considered in this field weree ither seasonality or replacing feature of the products. However, just one of them considered both of the features; in which price and order size for just one of the products were variable (given price and order size for the second product). Thus, in this paper, not only we considered seasonality and substitution, but also we determined prices and order quantities for both of the products, simultaneously.

7. REFERENCES

1. Zhang, C,T., Wang, H,X. and Ren, M,L," Research on pricing and coordination strategy of green supply chain under hybrid production mode", ***Computers and Industrial Engineering***, Vol. 72, (2014), 24-31.
2. Xie, Y., Tai, A,H. Ching, W,K. and Siu, T,K, " Pricing strategy for a two-echelon supply chain with optimized return effort level", ***International Journal of Production Economics***, Vol. 182, (2016), 185-195.
3. Seifbarghy, M., Kalaei, M. and Rasouli, N, " Supplier selection and order quantity assignment using conditional entropy and Chebyshev goal programming approach", 12th International Conference of Iranian Operations Research Society, (2019), 4337- 4350.
4. Li, Y., Zhang, S. and Han, J, "Dynamic pricing and periodic ordering for a stochastic inventory system with deteriorating items", ***Automatica***, Vol. 76, (2017), 200-213.
5. Shaikh, A.A., Khan, M.A.A., Panda, G.C. and Konstantaras, I, "Price discount facility in an EOQ model for deteriorating items with stock-dependent demand and partial backlogging", ***International Transactions in Operational Research***, Vol. 26, No. 4, (2019), 1365-1396.
6. Aviv, Y. and Pazgal, A," Optimal Pricing of Seasonal Products in the Presence of Forward-Looking Consumers and Prospects", ***Journal of Manufacturing and Service Operations Management***, Vol. 10, (2008), 339-359.

7. You, S.P and Chen, C.T, "Dynamic pricing of seasonal goods with spot and forward purchase demands", ***Journal of Computers and Mathematics with Applications***, Vol. 54, (2007), 490–498.

8. Chopra, S. and Meindle, P "Supply chain management Strategy, Planning, and Operations (3rd Ed)", New jersey: Prentice-Hall, (2007).

9. Transchel, S, " Inventory management under price-based and stockout-based substitution", ***European Journal of Operational Research***, Vol. 262, No. 3, (2017), 996-1008.

10. Kim, S. and Bell, P.C, "Optimal pricing and production decision in the presence of symmetrical and asymmetrical substitution", ***Omega***, Vol. 39, (2011), 528-538.

11. Rasouli, N. and Nakhai, I, "Joint Pricing and Inventory Control for Seasonal and Substitutable Goods Mentioning the Symmetrical and Asymmetrical Substitution", ***International Journal of Engineering Transaction C Aspects***, Vol. 27, No. 9, (2014), 1385-1392.

12. Wang, C., Huang, R. Pricing for seasonal deteriorating products with price- and ramp-type time-dependent demand. ***Computers & Industrial Engineering***, Vol. 77, (2014), 29-34.

13. Due, J., Zhang, J. and Hua, G, "Pricing and inventory management in the presence of strategic customers with risk preference and decreasing value", ***International Journal of Production Economics***, Vol. 164, (2015), 160-166.

14. Rabbani, M., Pourmohammad Zia, N. and, Rafiei, H, "Joint optimal dynamic pricing and replenishment policies for items with simultaneous quality and physical quantity deterioration", ***Applied Mathematics and Computation***, Vol. 287, (2016), 149-160.

15. Naimi Sadigh, A., Chaharsooghi, S. K. and Sheikhmohammady, M, "A game theoretic approach to coordination of pricing, advertising, and inventory decisions in a competitive supply chain", ***Journal of Industrial and Management Optimization***, Vol. 12, (2016), 337-355.

16. Aminnaseri, M. R. and Azari Khojasteh, M, "Price competition between two leader–follower supply chains with risk-averse retailers under demand uncertainty", ***The International Journal of Advanced Manufacturing Technology***, Vol. 79, (2015), 377-393.

17. Bian, W., Shang, J. and Zhang, J, "Two-way information sharing under supply chain competition", ***International Journal of Production Economics***, Vol. 178, (2016), 82-94.

18. Taleizadeh, A.A. and Rasouli-Baghban A, "Pricing and lot sizing of a decaying item under group dispatching with time-dependent demand and decay Rates", ***Scientia Iranica,*** Vol. 25, No. 3, (2018), 1656-1670.

19. Karakul, M. and Chan, L "Analytical and managerial implications of integrating product substitutability in the joint pricing and procurement problem", ***European Journal of Operational Research***, Vol. 190, (2008), 179–204.

20. Karakul, M. and Chan, L.M.A, "Joint pricing and procurement of substitutable products with random demands- A technical note", ***European Journal of Operational Research***, Vol. 201, (2010), 324-328.

21. Soleimani, F., Arshadikhamseh, A. and Naderi, B, "Optimal decisions in a dual-channel supply chain under simultaneous demand and production cost disruptions", ***Annals of Operations Research***, (2014), 1-21.

22. Dong, L., Kouvelis, P. and Tian, Z, "Dynamic pricing and inventory control of substitute products", ***Journal of manufacturing and service operations management***, Vol. 11, (2009), 317-339.

23. Noori-daryan M., Taleizadeh, A.A. and Jolai, F, "Analyzing pricing, promised delivery lead time, supplier-selection, and ordering decisions of a multi-national supply chain under uncertain environment", ***International Journal of Production Economics,*** Vol. 209, (2019), 236-248.

24. Moradinasab, N., Aminnaseri, M., Behbahani, T.J. and Jafarzadeh, H, "Competition and cooperation between supply chains in multi-objective petroleum green supply chain: A game theoretic approach", ***Journal of Cleaner Production***, Vol. 170, (2018), 818-841.

25. Elyasi, M., Jafarzadeh, H. and Khoshalhan, F, "An economical order quantity model for items with imperfect quality: A non-cooperative dynamic game theoretical model", 3rd International Logistics & Supply Chain Conference, (2012).

26. Frasi, E., Yousefi Yegane, B. and Moniri, A, "Simultaneous Pricing, Routing, and Inventory Control for Perishable Goods in a Two-echelon Supply Chain", ***International Journal of Engineering Transaction A Basics***, Vol. 31, No. 7, (2018), 1074-1081.

27. Fattahi, P., Tanhatalab, M. and Bashiri, M, "Bi-objectives Approach for a Multi-period Two Echelons Perishable Product Inventory-routing Problem with Production and Lateral Transshipment", ***International Journal of Engineering Transaction C Aspects,*** Vol. 30, No. 6, (2017), 876-886.

8. APPENDIX

Appendix A According to Equation (14), we have the following derivations for $i, w = 1{,}2,\ w \neq i,\ 1 \leq j, r \leq n,\ r \neq j$:

$$\frac{\partial^2 F}{\partial {p_{i,j}}^2} = -2M(\beta_i + L)$$

$$\frac{\partial^2 F}{\partial p_{i,j} p_{w,j}} = 2LM$$

$$\frac{\partial^2 F}{\partial p_{i,j} p_{i,r}} = 0$$

$$\frac{\partial^2 F}{\partial p_{i,j} p_{w,r}} = 0$$

Now, we can form the hessian matrix, which is (2n*2n) as follows:

$$\begin{bmatrix}
\frac{\partial^2 \pi}{\partial {p_{1,1}}^2} & \frac{\partial^2 \pi}{\partial p_{1,1} p_{1,2}} & \dots & \frac{\partial^2 \pi}{\partial p_{1,1} p_{1,\mathrm{n}}} & \frac{\partial^2 \pi}{\partial p_{1,1} p_{2,1}} & \dots & \frac{\partial^2 \pi}{\partial p_{1,1} p_{2,n}} \\
\frac{\partial^2 \pi}{\partial p_{1,2} p_{1,1}} & \frac{\partial^2 \pi}{\partial {p_{1,2}}^2} & \dots & \frac{\partial^2 \pi}{\partial p_{1,2} p_{1,\mathrm{n}}} & \frac{\partial^2 \pi}{\partial p_{1,2} p_{2,1}} & \dots & \frac{\partial^2 \pi}{\partial p_{1,2} p_{2,\mathrm{n}}} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\frac{\partial^2 \pi}{\partial p_{2,1} p_{1,1}} & \frac{\partial^2 \pi}{\partial p_{2,1} p_{1,2}} & \dots & \frac{\partial^2 \pi}{\partial p_{2,1} p_{1,\mathrm{n}}} & \frac{\partial^2 \pi}{\partial {p_{2,1}}^2} & \dots & \frac{\partial^2 \pi}{\partial p_{2,1} p_{2,\mathrm{n}}} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\frac{\partial^2 \pi}{\partial p_{2,\mathrm{n}} p_{1,1}} & \frac{\partial^2 \pi}{\partial p_{2,\mathrm{n}} p_{1,2}} & \dots & \frac{\partial^2 \pi}{\partial p_{2,\mathrm{n}} p_{1,\mathrm{n}}} & \frac{\partial^2 \pi}{\partial p_{2,\mathrm{n}} p_{2,1}} & \dots & \frac{\partial^2 \pi}{\partial {p_{2,\mathrm{n}}}^2}
\end{bmatrix}$$

The determinant of principle minors are as follows:

D_1: $(-1)^1\, 2M(\beta_1 + L) < 0$ D_2: $(-1)^2 4M^2(\beta_1 + L)^2 > 0$

D_3: $(-1)^3\, 8M^3(\beta_1 + L)^3 < 0$ …

D_{2n}: $(-1)^{2n}\,(2)^{2n}\, M^{2n}\,(L\beta_1 + L\beta_2 + \beta_1\beta_2)^n > 0$

It is clear that for all principle minors, $(-1)^k D_k > 0$, $k = 1,2,\dots,2n$; thus, the Hessian matrix is negative-definite and accordingly profit function (π) is a strictly concave function and the proof is complete.

Appendix B In this part, we estimate the demand parameters for ice cream in US, using data from 1973 to 2019 (June to August), derived by website: https://fred.stlouisfed.org. As mentioned before, these data are for different kinds of ice creams and frozen foods. Therefore, in order to separate the data for two kinds of ice cream, we assume that one of the ice creams consists 8% of total demand and the other one consists 10% of it. In addition, for prices we do the same with 10%, and 9%, respectively. According to the fact that the price of seasonal products decreases during the sale season [7, 11], we use 9 and 8% for the price of first product during July to August; 8 and 7% for the second product, during the same months. Moreover, the data of demand and price are based on the related data of 2012 and 1983 as indices, respectively. Hence, in order to normalize the data, we assume 10,000 for demand of 2012 and $90 for price of 1983. At last, in order to remove the inflation rate on prices, we consider 2019 as the decision year and change the prices of 1972 to 2018 based on 2019, using inflation rate of US repoerted in the website: https://www.in2013dollars.com[1].

First, we consider multiple linear regression and estimate the demand functions as bellow, by using Minitab 19:

$$d_{1,j} = 2244 - 62\, p_{1,j} - 5727(\, p_{1,j} - \, p_{2,j})$$
$$d_{2,j} = 1395 - 865\, p_{2,j} - 5727(\, p_{2,j} - \, p_{1,j})$$

We consider $d_{1,j}$ and $d_{2,j}$ as cumulative demand during period j, for products 1 and 2, respectively, $p_{1,j}$ as price of product 1 in period j, and $p_{2,j}$ as price of product 2 in period j. The parameters for $d_{1,j}$ ($d_{2,j}$) are estimated with R-sq=82.96% (70%). Since $d_{1,j}$ and $d_{2,j}$ are cumulative demand during period j, we have:

$$d_{i,j}=\int_{(j-1)M}^{jM} d_{t,i,j}\, dt = \frac{e^{-jMg_i}(-1+e^{Mg_i})\alpha_i}{g_i} + M(-\beta_i p_{i,j} - L(p_{i,j} - p_{w,j}))$$

Now, considering $M\beta_1$=62, $M\beta_2$= 865, ML =5727, $\frac{1}{\mathrm{n}}\sum_{j=1}^{n}(\frac{e^{-jMg_1}(-1+e^{Mg_1})\alpha_1}{g_1}) = 2224$, $\frac{1}{\mathrm{n}}\sum_{j=1}^{n}(\frac{e^{-jMg_2}(-1+e^{Mg_2})\alpha_2}{g_2})$ =1395, and by assuming $g_1 = g_2 = 0.001$, we have:

$$M=\frac{T}{n} = \frac{June\ to\ August\ (3\ months)}{3} = \frac{3\ (months)*30\ (days)*8\ (hours)}{3} = 240$$

$\beta_1 = 0.26$, $\beta_2 = 3.6$, L=23.86, $\alpha_1 = 13.6$, $\alpha_2 = 8.47$

Coordination of Pricing and Order Quantity for Two Replaceable and Seasonal Products

N. Rasouli[a,b], M. R. Amin-Naseri[a], I. Nakhai Kamalabadi[c], A. Hosseinzadeh Kashan[a]

[a] *Department of Industrial and System Engineering, Tabriat Modares University, Iran*
[b] *Group for Research in Decision Analysis (GERAD), HEC Montreal University, Canada*
[c] *Department of Industrial Engineering, University of Kurdistan, Kurdistan, Iran*

P A P E R I N F O

Paper history:
Received 26 November 2019
Received in revised form 16 December 2019
Accepted 22 Januray 2020

Keywords:
Pricing
Order Quantity
Seasonal Products
Substitution
Karush Kuhn Tucker Conditions

چکیده

در این مقاله، مسئله قیمت گذاری و تعیین میزان سفارش برای دو کالای فصلی که قابلیت جایگزینی دارند ارائه می شود. فرض ما برا این است که مشتریان حساس به قیمت هستند و تمایل به خرید کالای ارزان تر را دارند، که این منجر به جایگزینی یک طرفه برا پایه مشتری و بر اساس قیمت می‌گردد. ابتدا، مدل‌سازی ریاضی برای یک بنگاه با دو کالای فصلی و جایگزین توسعه داده می شود. مدل به دنبال ماکزیمم سازی سود توسط تعیین قیمت، میزان سفارش و تعداد دوره برای هر دو کالاست. سپس تقعر تابع هدف و اینکه دارای نقطه ماکزیمم هست، اثبات می‌گردد. در ادامه، یک الگوریتم دقیق برا پایه شرایط کاروش کان تاکر، برای تعیین متغیرهای تصمیم ارائه می شود و در انتها، مثال عددی و آنالیز حساسیت برای نشان دادن کارایی روش حل و الگوریتم ارائه شده، توسعه داده می‌شود.

doi:*10.5829/ije.2020.33.03c.08*

[1] Some of the data were outbound that we removed them in order to get a more appropriate model.